THE UNIVERSITY OF ALBERTA

AN EMPIRICAL AND THEORETICAL STUDY OF FACTORIAL INVARIANCE

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ABSTRACT

The purpose of this paper is to report an empirical and theoretical study of factorial invariance. A factor is considered invariant if it can be shown to be independent of both the sample of individuals tested and the sample of variables used in the analysis. Defining invariance in this way poses two problems: designing techniques for assessing invariance and determining what kind of meaning should be attributed to an invariant factor.

In order to shed some light on these questions two matrices of correlations based on two independent determinations of the photopic visibility curve were factored and rotated to oblique simple structure. The first matrix yielded red, green, and blue factors and the second matrix yielded red, green, yellow, and blue factors. Kaiser's and Wrigley's methods of relating factors were applied in an attempt to determine the degree of similarity of the two sets of factors. The results indicated that the red, green, and blue factors from the two studies were very similar. Kaiser's method of relating factors is probably superior since it is more objective than Wrigley's method.

When the factors were plotted against the spectrum the resulting curves appeared similar to curves that have been proposed by physiologists to account for the action of color receptors. This suggests that invariant factors may be interpreted as hypothetical constructs.

The author suggests that an invariant factor matrix be defined as

$$\mathcal{U} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} V_i$$

where the V_i refers to independently rotated factor matrices. From this it seems to follow that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} T_i \longrightarrow I$$

where T_i is a transformation that carries each V_i into the best average of them all; must hold if $\mathcal{U}$ is to be unique.

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CONTENTS

The Problem

The purpose of this paper is to report an empirical and theoretical study of factorial invariance. Factorial invariance refers to the degree of similarity of factorial results when analyses are performed on different samples of individuals using different samples of variables. If factor analysis is considered as a technique for identifying psychological variables which are then to be studied by other techniques, factorial invariance becomes one of the most important problems of contemporary factor theory; for unless a factor is independent of both the sample of individuals tested and the sample of variables used in the analysis it can not be considered as an independent psychological variable.

Factorial invariance apparently involves two problems. First there is the problem of designing techniques for assessing the degree of factorial invariance. Then there is also the question of what meaning should be attributed to invariant factors. Should they be considered as some type of descriptive measure, such as a standard deviation, or do they possess the status of intervening variables or hypothetical constructs?

Specifically, then, an attempt will be made to empirically examine these questions. First, two independent sets of data from the same domain will be factored and rotated to oblique simple structure. Then several of the better techniques for assessing factorial invariance will be applied in an attempt to determine if the factors possess a considerable degree of in-

variance. And finally, if the factors seem invariant an attempt will be made to determine what kind of construct they represent.

After considering several sets of data, two matrices of correlations (Jones, 1948; Jones, 1950), based on two independent determinations of the photopic visibility curve were chosen. These psychophysical data were chosen over other purely psychological data chiefly because considerable non-factorial knowledge is available concerning the psychophysiology of vision. This, it was hoped, would make possible an attempt to relate factors to other non-factorial knowledge and thereby throw some light on the nature of invariant factors.

Techniques for Assessing Invariance

In factor analysis as in other statistical techniques a sample is taken from a universe, measurements are made on the individuals, these measurements are treated according to a mathematical model and then an attempt is made to make inferences about the universe that was sampled. In factor analysis one is dealing with two universes: (1) a universe of individuals from which the sample of individuals who take the tests is drawn, and (2) a universe of psychological content from which the sample of variables used in the analysis is drawn. This leads to the four special cases of factorial invariance listed in the table below. Hunka (1962) has listed the procedures which seem most appropriate for each case.

		Psychological Content	
		same variables	different variables
INDIVIDUALS	Same Individuals	(1) Kaiser 1960a Wrigley-Neuhaus 1955	(2) Kaiser 1960b Tucker 1958 Wrigley-Neuhaus 1955
INDIVIDUALS	Different Individuals	(3) Ahmavaara 1954 Burt (1948) Kaiser (1960 b) Tucker (1951) Cattell (1949 and 1960)	(4) No adequate solution

Of the four special cases mentioned in the table most attention has been paid to (2) different variables - same individuals, and (3) same variables - different samples. The reason is that the first case, same individuals - same variables, or factor reliability, is not very interesting because it lacks generality; and the fourth case, different variables - different samples, though it is most general and therefore most interesting seems to have no adequate objective solution. In this last instance a factor is independent of both the sample of individuals tested and the sample of variables used in the analysis and can surely be regarded as a significant independent psychological variable.

Cattell (1962) has listed six sources of evidence that can be used to assess the degree of invariance of a simple structure factor. They are:

> "(1) The nature of the factor pattern of loadings, (or some related dimension variable relation profile), or some function thereof.
>
> (2) The agreement of the set of correlations of the given factor, in its own matrix, with other (different, already identified) factors in that matrix, with the set known to be typical for that factor, from other matrices.
>
> (3) The absolute size of the mean variance contribution of the given factor in relation to a standard, stratified sample of variables.
>
> (4) Regard for "proof by elimination". This requires that every well designated factor experiment should, regularly and systematically,

include among the variables introduced to study its own special research topic, or domain, <u>a context of marker variables</u>, chosen to represent comprehensively the well known established factors.

(5) The behavior of the factor as a psychological entity, treated as an independent or dependent scientific variable, when involved in subsequent accompanying manipulative experiment.

(6) Regard for the character of the modifications in (1), (2), (3), and (4) above that would be predicted from (a) sampling error, (b) real differences in population (selection), (c) changing reliabilities of variables, (d) changing conditions of administration of tests, and (e) differences of experimental design, as between R-, P-, and incremental R- techniques."

Although all these sources of evidence need to be considered in order to establish the degree of invariance of a factor, (1) and (5) are obviously the most important. For unless two factors have approximately the same loading profile they are not the same factor. And even though all other evidence points to invariance, if a factor does not behave as a stable psychological variable in other types of experiments it could not be considered as an invariant factor. For these reasons this paper will be primarily concerned with (1) comparisons of profiles of loadings, and (5) looking at the factors as psychological constructs.

Since factor loadings are proportional to the standard

deviation of the tests they represent, (Henrysson, 1957), one should compare only analyses based on covariance or loadings corrected for variance difference (Tucker, 1951). If different studies are to be compared the factor loadings should be estimated by the maximum likelihood method since loadings determined in this manner are independent of the units of measurement of the tests.

However, Tucker (1951) has pointed out that if the analyses were carried out on correlations, an adjustment to the rows of the factor matrices can make them comparable. He recommends that $\underline{A}$, the factor matrix for the first study, be pre-multiplied by a diagonal matrix $\underline{d}_{jA}$ defined as

$$\underline{d}_{jA} = \sigma_{jA}/[1/2(\sigma_{jA} + \sigma_{jB})] \qquad (1)$$

and $\underline{B}$, the factor matrix for the second study, be premultiplied by $\underline{d}_{jB}$ defined as

$$\underline{d}_{jB} = \sigma_{jB}/[1/2(\sigma_{jA} + \sigma_{jB})] \qquad (2)$$

where σ_{jA} and σ_{jB} represent, respectively, the standard deviations of the variables from study A and study B. However, these considerations are not so important if the studies have been rotated to simple structure which is independent of the

units of measurement of the tests (Thomson, 1951).

The most common method for attempting to assess factorial invariance has been the use of some form of correlation. However, as Barrow and Burt (1954) have pointed out, the ordinary correlation cannot be used to assess factorial invariance because it is based on deviations from the mean. For example, if the correlation coefficient were calculated for two factors that were very similar except that one had slightly higher (than its mean) loadings on one end and the other had slightly higher (than its mean) loadings on the other end, then the correlation between the loadings would be low negative. Yet these factors could really be very similar. In order to assess the degree of similarity between two sets of loadings the degree of proportionality between the sets of factor scores or measurements must be determined. This requires the calculation of the unadjusted (for means) coefficient of correlation.

The most common correlational index of factorial invariance is the Burt (1948)-, Tucker (1951)-, Wrigley (1955) procedure. They propose that in case (2), same individuals-different variables, that the scalar products of the vectors of factor scores be used as an index of factorial invariance.

The index can be calculated from the following formula:

$$\ell_{jk} \qquad \frac{\sum_{i=1}^{n} p_{ji}\, q_{ki}}{\sqrt{\sum_{i=1}^{n} p_{ji}^{2} \sum_{i=1}^{n} q_{ki}^{2}}} \qquad (3)$$

where

ℓ_{jk} = index of similarity for factor j of first study and factor k of second study.

p_{ji} = factor score of person i in factor j of first study

q_{ki} = factor score of person i in factor k of second study.

The index is a measure of the proportionality of the two vectors of factor scores. In geometric terms it give the cosine of the angle between the two normalized vectors of factor scroes. The index ranges through $\pm$ 1.00; when it is near to either + 1.00 or - 1.00 the factor loading profiles may be considered identical, when it is near zero they may be considered orthogonal. Since this method is applicable whenever we have the same individuals it can also be used for case (1), same individuals - same variables.

Burt (1948), Tucker (1951) and Wrigley (1955) have attempted

to apply this technique to case (3), same variables - different samples. In this case the index takes the forms:

$$\ell_{jk} = \frac{\sum_{i=1}^{p} f_{ij}\, g_{ik}}{\sqrt{\sum_{i=1}^{p} f_{ij}^2 \cdot \sum_{i=1}^{p} g_{ik}^2}} \tag{4}$$

where

ℓ_{jk} = the index of invariance for factor j of study one and factor k of study two.

f_{ij} = the ith loading of the jth factor of study one.

g_{ik} = the ith loading of the kth factor of study two.

Here the index gives the angle between the two vectors of factor loadings.

But Kaiser (1960b) has pointed out several difficulties in this case. The chief difficulty is that one must assume that a column of factor loadings defines a factor. However, this is not true, though a row of a factor matrix defines a variable in terms of a linear combination of factor loadings, a column of a factor matrix does not define a factor in terms of a linear combination of variables. It can be shown (Harman 1960) that the coefficients for the linear combination of tests that define a factor are $\underline{b}'\underline{R}^{-1}$, where $\underline{b}$ is a vector of primary factor

structure and $\underline{R}^{-1}$ is the inverse of the original correlation matrix.

Kaiser (1960b) has attempted to rationalize ℓ_{jk} , here written in vector notations for greater simplicity

$$\ell_{12} = \frac{\underline{b}_1' \ \underline{b}_2}{\sqrt{\underline{b}_1' \ \underline{b}_1 \qquad \underline{b}_2' \ \underline{b}_2}} \tag{5}$$

where $\underline{b}_1$ and $\underline{b}_2$ are any vectors of factor loadings from study one and two.

The factor or its estimate from study one is given by $\underline{b}_1' \ \underline{R}_{11}^{-1} \ \underline{Z}_1$, and from study two by $\underline{b}_2' \ \underline{R}_{22}^{-1} \ \underline{Z}_2$. The index now becomes

$$\ell_{12} = \frac{\underline{b}_1' \ \underline{R}_{11}^{-1} \ \underline{R}_{12} \ \underline{R}_{22}^{-1} \ \underline{b}_2}{\sqrt{\underline{b}_1' \ \underline{R}_{11}^{-1} \ \underline{b}_1 \qquad \underline{b}_2' \ \underline{R}_{22}^{-1} \ \underline{b}_2}} \tag{6}$$

where

$\underline{b}_1$ and $\underline{b}_2$ = nxl vectors of primary factor structure from studies one and two respectively.

$\underline{R}_{11}^{-1}$ and $\underline{R}_{22}^{-1}$ = respectively the inverses of the correlation matrice of studies one and two.

$\underline{R}_{12}$ = the $n_1 \times n_2$ cross correlation matrix between the two studies.

But $\underline{R}_{12}$ is the crucial missing element in this case of different individuals. If we assume only the same variables and than $n_1 = n_2 = n$, and if $\underline{R}_{11} = \underline{R}_{22} = \underline{R}_{12}$ (approximately in samples, exactly in populations) then the index reduces to

$$\ell_{12} = \frac{\underline{b}_1' \underline{R}^{-1} \underline{b}_2}{\sqrt{\underline{b}_1' \underline{R}^{-1} \underline{b}_2 \quad \underline{b}_2' \underline{R}^{-I} \underline{b}_2}} \qquad (7)$$

Now only if $\underline{R}^{-1} = \underline{I}$ would $\underline{l}_{12}$, the Burt-Tucker-Wrigley-Neuhaus coefficient, reduce to (6), the real correlation between the factors.

However, if $\underline{R}^{-1}$ were scalar, then $\underline{b'R}^{-1}$ would be proportional to $\underline{b}$; and one could make an approximate interpretation. Guttman (1940 and 1953) has shown that for factor analytic procedures to have validity for drawing inferences from a sample of variables to a universe of psychological content it is sufficient that $\underline{R}^{-1}$ become diagonal as n approaches infinity. For a finite sample $\underline{R}^{-1}$ should be nearly diagonal, but even this will not occur unless the variables are a comprehensive sample of the universe of psychological content. Now if these conditions have been met $\underline{R}^{-1}$ can become scalar only if $\underline{R}=\underline{I}$; when there are no factors in the usual sense. Thus ℓ_{12} will always be distorted because $\underline{R}^{-1}$ will never be exactly scalar.

This index compares factors as they stand, but it is possible that some other slightly different rotation might give a higher index of similarity. Both Tucker (1951) and Wrigley-Neuhaus (1955) have proposed transformations that will ensure that the index is a maximum.

Wrigley-Neuhaus (1955) propose to find orthogonal transformations $\underline{T}_1$ and $\underline{T}_2$ of each factor analysis $\underline{F}_1$ and $\underline{F}_2$ so that each factor loading vector of study one is orthogonal to all but one factor vector of study two. They show that $\underline{T}_1$ is the matrix of latent vectors of $(F_1' \; F_2) \; (F_1' \; F_2)'$ and $\underline{T}_2$ is the matrix of latent vectors of $(F_2' \; F_1) \; (F_2' \; F_1)'$. The index is then computed on the columns of $\underline{F}_1 \; \underline{T}_1$ and $\underline{F}_2 \; \underline{T}_2$. If both factor analyses were performed on the same individuals and factor scores were compared this procedure would be equivalent to computing the canonical correlation for the two sets of variables (Wrigley 1955).

Tucker uses a similar procedure though his goal is somewhat different. While Wrigley-Neuhaus (1955) are concerned with the degree of similarity between two sets of factors, Tucker (1951) is concerned with factor matching, i.e. finding two sets of factors which may be considered essentially the same.

In order to achieve maximum congruence he wishes to maximize the projection of factor one in study one on factor one of study two. This can be achieved by minimizing

$$g_r = \frac{\sum_{i=1}^{p} (f_{ij} - g_{ik})^2}{\sum_{i=1}^{p} (f_{ij} + g_{ik})^2} \tag{8}$$

when f_{ij} is the ith loading on the jth factor of study one, and g_{ik} is the ith loading on the kth factor of study two. g_r may be considered to be a measure of the extent to which the factors from the two studies can be replaced by their average.

He shows that when g_r is minimized

$$\ell_{jk} = \frac{\sum_{i=1}^{p} f_{ij} \quad g_{ik}}{\sqrt{\sum_{i=1}^{p} f_{ij}^2 \cdot \sum_{i=1}^{p} g_{ik}^2}} \tag{9}$$

his proposed measure of congruence is maximized and that they are related by the relation

$$g_r = \frac{1 - \ell_{jk}}{1 + \ell_{jk}} \tag{10}$$

In order to get maximum congruence he applies the following transformations to the factors from the two studies.

$$\underline{T}_{mrA} = \underline{M}_{mpA} \quad \underline{B}_{pA}^{-1/2} \; \underline{M}_{rA} \; \underline{d}_{r} \tag{11}$$

$$\underline{T}_{mrB} = \underline{M}_{mpB} \quad \underline{B}_{pA}^{-1/2} \; \underline{M}_{rB} \; \underline{d}_{r} \tag{12}$$

where

$\underline{M}_{mpA}$ = eigenvectors of $\underline{f}'\underline{f}$

$\underline{M}_{mpB}$ = eigenvectors of $\underline{g}'\underline{g}$

$\underline{B}_{pA}^{-1/2}$ = the square roots of the inverse of the eigenvalues of $\underline{f}'\underline{f}$

$\underline{B}_{pB}^{-1/2}$ = the square root of the inverse of the eigenvalues of $\underline{g}'\underline{g}$

$\underline{M}_{rA}$ = eigenvectors of $(\underline{B}_{pA}^{-1/2} \; \underline{f}' \; \underline{g} \; \underline{B}_{pB}^{-1/2}) (\underline{B}_{pA}^{-1/2} \; \underline{f}' \; \underline{g} \; \underline{B}_{pB}^{-1/2})'$

$\underline{M}_{rB}$ = eigenvectors of $(\underline{B}_{pA}^{-1/2} \; \underline{f}' \; \underline{g} \; \underline{B}_{pB}^{-1/2})' \; (\underline{B}_{pA}^{-1/2} \; \underline{f}' \; \underline{g} \; \underline{B}_{pB}^{-1/2})$

$\underline{d}_{r}$ = a constant that will on the average normalize both tranformation matrices.

He discards the factors that have a low $\ell_{\cdot jk}$ value and rotates the remaining ones for each study to simple structure.

The Tucker (1951) and Wrigley-Neuhaus (1955) methods seem very similar. They put both sets of factors into the same space and use some method of maximizing the projections of the factors they wish to compare. However, Kaiser (1960) reverses this procedure. He puts the two sets of test vectors into the same space, rotates them for maximum overlap and then determines the angles between the factor vectors. The cosines of these angles are interpreted as correlations between the factors from the studies.

He post-multiplies $\underline{F}_2$ by a transformation $\underline{K}$ such that the test vectors of $\underline{F}_2$ will have maximums projection on the test vectors of the $\underline{F}_1$. He defines

$$\underline{P} = (\underline{F}_1' \underline{F}_2) \quad (\underline{F}_1' \underline{F}_2)'$$

where $\underline{F}_1$ and $\underline{F}_2$ are any orthogonal factorings of $\underline{R}_1$ and $\underline{R}_2$, the correlation matrices for the two studies. The eigenvectors and eigenvalues of $\underline{P}$ are then determined.

$$\underline{P} = (\underline{F}_1' \underline{F}_2) \quad (\underline{F}_1' \underline{F}_2)'. \tag{14}$$

$$= \underline{U}\ \underline{M}\ \underline{U}' \tag{15}$$

$$= \underline{U}\ \underline{M}^{1/2} \underline{U}'\ \underline{U}\ \underline{M}^{1/2} \underline{U}' \tag{16}$$

Since $\underline{U}\ \underline{M}^{1/2}\underline{U}'$ provides the conditions for maximizing test vector projections we have

$$\underline{F}_1'\ \underline{F}_2\ \underline{K} = \underline{U}\ \underline{M}^{1/2}\underline{U}' \qquad (17)$$

Solving

$$\underline{K} = [(\underline{F}_1'\ \underline{F}_2)'\ (\underline{F}_1'\ \underline{F}_2)]^{-1}\ (\underline{F}_1'\ \underline{F}_2)'\ \underline{U}\ \underline{M}^{1/2}\underline{U}' \qquad (18)$$

Kaiser is not particularly interested in the transformed factors, but in $T_2'K$ and its relation to $\underline{T}_1'$, where $\underline{T}_1'$ and $\underline{T}_2'$ are transformation matrices that rotate $\underline{F}_1$ and $\underline{F}_2$ to simple structure on the primary factors. The cosines between the factors of the two studies is then given by

$$\begin{aligned} \underline{L}_{12} &= \underline{T}_1'\ (\underline{T}_2'\ K)' \\ &= \underline{T}_1'\ \underline{K}'\ \underline{T}_2 \end{aligned} \qquad (19)$$

More generally the cosines between all r_1 factors of study one and all r_2 factors of study two are given by

$$\begin{bmatrix} \underline{L}_{11} & \underline{L}_{12} \\ \underline{L}_{21} & \underline{L}_{22} \end{bmatrix} = \begin{bmatrix} \underline{T}_1'\ \underline{T}_1 & \underline{T}_1'\ \underline{K}'\ \underline{T}_2 \\ \underline{T}_2'\ \underline{KT}_1 & \underline{T}_2'\ \underline{T}_2 \end{bmatrix} \qquad (20)$$

Where $\underline{L}_{11}$ and $\underline{L}_{22}$ are correlations between factors within studies

and $\underline{L}_{12}$ and its transpose $\underline{L}_{21}$ give the proposed measure of factor similarity.

Since the method is based on the rotation of communality vectors[1] rather than test vectors it should probably not be applied when one study has many more factors than the other. In this case the communality vectors, even though based on the same variables, could differ appreciably, because of the possibility that the greater number of factors could convert specific variance into common variance. A related difficulty is that the longer communality vectors will have more weight and will therefore bias the rotation to maximum overlap. One way to avoid this difficulty would be to normalize the rows of both factor matrices.

One may desire a measure of the closeness of the two sets of test vectors after the transformation $\underline{K}$ has been applied. The cosines between the N pairs is given by the diagonal of the matrix.

$$A = \underline{H}_1^{-1} \underline{F}_1 \underline{K}' \underline{F}_2 \underline{H}_2^{-1} \qquad (21)$$

where $\underline{H}_1$ and $\underline{H}_2$ are diagonal matrices of the square roots of the

1. A communality vector refers to a vector that represents the common part of a test.

communalities of the tests from the two studies. Inspection of these values can help determine if any pair of tests was a poor match. A mean measure of the fit can also be defined

$$\underline{S} = \frac{1}{n} \text{trace} [\underline{H}_1^{-1} \underline{F}_1 \underline{K} \underline{F}_2' \underline{H}_2^{-1}] \qquad (22)$$

$\underline{S}$ varies between zero and one and could be interpreted as an average of the correlations between the n pairs of tests.

Ahmavaara (1954) provides what is probably the simplest method of assessing invariance. Given two sets of factors $\underline{F}_1$ and $\underline{F}_2$ he wishes to find a transformation $\underline{X}_{12}$ that will carry $\underline{F}_1$ into $\underline{F}_2$.

$$\underline{F}_1 \underline{X}_{12} = \underline{F}_2 \qquad (23)$$

solving for $\underline{X}_{12}$ we get

$$\underline{F}_1 \underline{X}_{12} = \underline{F}_2$$

$$\underline{F}_1' \underline{F}_1 \underline{X}_{12} = \underline{F}_1' \underline{F}_2$$

$$\underline{X}_{12} = (\underline{F}_1' \underline{F}_1)^{-1} (\underline{F}_1' \underline{F}_2) \qquad (24)$$

The rows of $\underline{X}_{12}$ are normalized to give the proposed measure of factorial invariance.

Another transformation $\underline{X}_{21}$ which carries $\underline{F}_2$ into $\underline{F}_1$ may be found

$$\underline{F}_2 \, \underline{X}_{21} = \underline{F}_1$$

$$\underline{X}_{21} = (\underline{F}_2' \, \underline{F}_2)^{-1} \, (\underline{F}_2' \, \underline{F}_1) \qquad (25)$$

Obviously $\underline{X}_{12}$ is not equal to $\underline{X}_{21}$ and a change of subscripts could give a different result!

In the case of oblique factors Ahmavaara wants to compare pattern on primaries for the studies concerned. He proceeds in the following manner:

$$\underline{X}_{12} = (\underline{V}_1' \, \underline{V}_1)^{-1} \, \underline{V}_1' \, \underline{F}_2 \qquad (26)$$

where

$\underline{X}_{12}'$ = the relationship between primary factors of the first study and the unrotated factors of the second study.

$\underline{V}_1$ = matrix of pattern on primaries for the first study

$\underline{F}_2$ = unrotated factor matrix for the second study.

The measure of invariance is then given by

$$\underline{X}'_{12} = \underline{X}'_{12} \; (\underline{\lambda}_2 \; \underline{D}_2^{-1}) \tag{27}$$

where $(\underline{\lambda}_2 \; \underline{D}_2^{-1})$ is the transformation matrix that carries $\underline{F}_2$ into $\underline{V}_2$, the pattern on primary for study two.

Ahmavaara's transformations $\underline{X}_{12}$ and $\underline{X}_{21}$ carry one factor space into another, but since transformations are defined only between subspaces of the same space, where a subspace may be the total space, this procedure is not mathematically defined. In order for his transformations to be mathematically legitimate he would have to assume that both factor spaces were subspaces of another space of at least r dimensions, where r is the dimensionality of the largest factor space. If it were possible to define a "domain space" the problems of factorial invariance would be greatly simplified.

Kaiser (1960) and Bargmann (1960) severely criticize Ahmavaara on a number of points. Kaiser believes that his procedure is only defined for different orthogonal rotations of the same original unrotated factor matrix. In this case he shows that

$$\underline{L}_{12} = \underline{L}_{11} \; \underline{X}_{12} \tag{28}$$

and

$$\underline{L}_{21} = \underline{L}_{22}\,\underline{X}_{21} \qquad (29)$$

where

$\underline{L}_{12}$ and $\underline{L}_{21}$ = respectively the correlations between the factors of the two different rotations

$\underline{L}_{11}$ and $\underline{L}_{22}$ = respectively the correlations of the factors within rotations one and two.

Thus $\underline{X}_{12}$ will give the correlations between the factors of the two different rotations when the correlations within the first study are zero and $\underline{X}_{21}$ will give the correlations for the different rotations when the correlations within the second rotation are zero.

And, according to Kaiser, in the case of two oblique rotations from the same original arbitrary factor matrix

$$\underline{X}_{12} = \underline{L}_{11}^{-1}\,\underline{X}'_{21}\,\underline{L}_{22} \qquad (30)$$

$$\underline{X}_{21} = \underline{L}_{22}^{-1}\,\underline{X}_{12}\,\underline{L}_{11} \qquad (31)$$

Thus $\underline{X}_{12}$ and $\underline{X}_{21}$ have no readily apparent relation to each other. Since one generally does not wish to make the restrictive assumption that the two factor matrices to be compared are merely different rotations of the same factor matrix, Kaiser[2] is probably correct when he calls Ahmavaara's method "hopelessly naive".

Tucker (1958) has devised what he calls an interbattery method of factor analysis which may be used to study invariance when the sample of individuals is the same, but the tests are different.[3] Given two test batteries administered to the same individuals the complete set of correlations between all tests may be represented by a super matrix.

$$\underline{R} = \begin{matrix} \underline{R}_{11} & \underline{R}_{21} \\ \underline{R}_{12} & \underline{R}_{22} \end{matrix} \qquad (32)$$

when $\underline{R}_{11}$ and $\underline{R}_{22}$ are the intercorrelations within the two batteries and $\underline{R}_{12}$ and its transpose are the intercorrelations between the tests from the two batteries. Using the fundamental

2. Personal communication (1962)

3. Werdelin (1963a) has recently developed another interbattery method. It seems to be similar to a special case of the Wrigley-Neuhaus (1955) method which exists when we have the same individuals, different tests and do not use a transformation to maximize the relationship between the two sets of factors.

equation of factor analysis

$$\begin{bmatrix} \underline{R}_{11} & \underline{R}_{21} \\ \underline{R}_{12} & \underline{R}_{22} \end{bmatrix} = \begin{bmatrix} \underline{A}_1 & \underline{G}_1 & \underline{0} \\ \underline{A}_2 & \underline{0} & \underline{G}_2 \end{bmatrix} \begin{bmatrix} \underline{A}_1' & \underline{A}_2' \\ \underline{G}_1' & \underline{0} \\ \underline{0} & \underline{G}_2' \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} \underline{A}_1 \underline{A}_1' + \underline{G}_1 \underline{G}_1', & \underline{A}_1 \underline{A}_2' \\ \underline{A}_2 \underline{A}_1', & \underline{A}_2 \underline{A}_2' + \underline{G}_2 \underline{G}_2' \end{bmatrix}$$

Where $\underline{A}_1$ and $\underline{A}_2$ are factors common to the two batteries, though not necessarily the same, and $\underline{G}_1$ and $\underline{G}_2$ are matrices of uniquenesses for the two batteries.

He shows that both $\underline{A}_1$ and $\underline{A}_2$ may be determined from $\underline{R}_{12}$ alone:

$$\underline{A}_1 = \underline{W}\,\underline{M}^{\frac{1}{2}} \tag{34}$$

$$\underline{A}_2 = \underline{V}\,\underline{M}^{\frac{1}{2}} \tag{35}$$

where

$\underline{M}^{\frac{1}{2}}$ = eigenvalues of $\underline{R}_{12}\,\underline{R}_{12}'$ or $\underline{R}_{12}'\,\underline{R}_{12}$

$\underline{W}$ = eigenvectors of $\underline{R}_{12}\,\underline{R}_{12}'$

$\underline{V}$ = eigenvectors of $\underline{R}_{12}'\,\underline{R}_{12}$

$\underline{A}_1$ and $\underline{A}_2$ may be independently rotated to simple structure.

He has also devised statistical tests for the residuals and for the reliability of the factors, but does not have much faith in them until more empirical evidence is available that can support their usefulness.

Cattell (1949, 1960) has developed two techniques for assessing invariance. He has suggested that r_p, his coefficient of pattern similarity, could be used to asses factor similarity. It is given by the formula

$$r_p = \frac{2K - \sum_{i=1}^{n} d_i^2}{2K + \sum_{i=1}^{n} d_i^2} \tag{38}$$

where

r_p = proposed index of similarity

K = median for X^2 on a sample of size n

d_i^2 = squared differences in factor loadings

n = number of common tests

There are at present several difficulties with this index. Since it is based on chi square it is not applicable to oblique factors. At present no transformation such as the Wrigley-Neuhaus (1955) or Kaiser (1960) is available to produce maximum relationship; however the Wrigley-Neuhaus or Kaiser transformation could possibly be used.

Cattell (1960) has also developed a nonparametric index, "The salient variable similarity index for factor matching", which he uses to assess factorial invariance. He divides the loadings for the common tests for both studies into salient and non salient, then calculates the probability of the number of similar loadings on the salient variables that occurred. If they could not have occurred by chance through sampling fluctuations the factors are considered similar. There are again several difficulties here. First it is nonparametric and therefore does not use much of the information in the factor matrices. Secondly it can only classify as similar - non similar; this is obviously not very useful.

In his latest article on the subject of invariance, Cattell (1962) specifies the six different possible factor matrices, e.g. pattern on primary, structure on reference, and discusses which of these should be used when attempting to assess invariance. He concludes that a factor matrix from which the effect of second order factors has been removed is the one that will give the best results. However, he does not show how to calculate it.

Now that the different techniques for assessing the similarity of factor loadings have been reviewed, an attempt must

be made to determine which is the most useful. The ideal invariance technique should possess the following characteristics:

(1) It should give an index varying through $\pm$ 1 so that it will be easily interpretable. Cattell's "Salient variable similarity index" (1960) does not possess this characteristic.

(2) The index should be applicable to both orthogonal and oblique factors. Cattell's "Coefficient of pattern similarity" (1949) does not possess this characteristic.

(3) A statistical test should be available to determine if the obtained value is significant. None of the presently available techniques possess this characteristic. However, there is some possibility that a statistical test could be devised for Tucker's "Inter-battery mathod of factor analysis" (1958) and for the Wrigley-Neuhaus (1955) method of assessing factor similarity since both these methods are applicable when the same individuals have been tested.

(4) Transformations should be available which make the factors maximally comparable without destroying the simple structure. Simple structure will not be destroyed if the

transformation is small in the sense that it performs a rotation through a small angle in order to achieve maximum similarity. Kaiser's (1960) transformation comes closest to meeting this characteristic.

None of the techniques reviewed possesses all these characteristics. However, for case (3) of different samples - same variables, Kaiser's (1960) method seems best, with Wrigley-Neuhaus' (1955) procedure running second. For case (2) of same individuals - different variables either Tucker's interbattery method or Wrigley-Neuhaus method of factor matching are applicable.

A solution to case (4) different individuals - different variables, will be difficult. At present the only available procedure would be to first establish the factorial similarity of the two sets of variables using a procedure such as Tucker's (1958) interbattery method. Then Kaiser's or Wrigley's method could be used to determine if the factors are invariant over different samples of individuals.

Color Vision and Factor Analysis

The several investigators who have applied factor analysis to color vision data have assumed that it can give evidence that can bear on the question of how many receptors are involved in color perception. If invariant factors of color vision could be established then there would be some justification for assuming that these factors are isomorphic to the entities or processes that mediate color perception.

Of the eight factor analytic studies of color vision only those of Jones (1954 and 1950), and Ekman (1954) involved rotation to simple structure. In each of these studies evidence was found for a blue, a yellow, and a red factor.

In each of these studies a different method of collecting the data was used. In his first study Jones used the data of Coblentz and Emerson (1918) who used the method of flicker spectrophotometry to determine the average photopic visibility curve. In his second study he used the data of Gibson and Tyndall (1923) who used the cascade method of spectrophotometry in order to determine the average photopic visibility curve. In both cases the reciprocal of the energy required to reach the absolute threshold was taken as a measure of sensitivity and intercorrelated using Pearson product moment correlations. In both studies red, yellow, and blue factors were found.

Ekman (1954) used still a different method. His subjects rated the similarity on a five point scale, of all possible paired comparisons of fourteen colored stimuli differing only in hue. These values were then converted to a zero to one scale and factored as correlations. Five factors violet, blue, green, yellow and red were extracted and rotated to orthogonal simple structure.

Now what can be said about the invariance of the blue, yellow and red factors which are common to all three studies? For greater clarity of exposition the loadings have been plotted against the spectrum in figures 1, 2, and 3.

Inspection of figure 1 indicates that there is considerable evidence for the invariance of the blue factor. All three curves have the same general shape and the same maximum. If it were not for the negative swing of Jones' (1948) blue factor in the yellow-green part of the spectrum there would be little doubt of the invariance of this factor. The reason for this swing is not clear at present. One possibility might be that this blue factor is really an improperly rotated bipolar yellow-blue factor. The negative swing might also be peculiar to this set of data for some unknown reason Still another possibility is that the factor is an incompletely rotated unipolar blue factor.

The evidence for the invariance of the yellow factor (see fig. 2) is not very convincing. There is some overlap between the curves, but no two have the same maximum.

From Figure 3 we see both Jones' red factors are very similar. But they are bipolar and Ekman's is not. If we consider only the positive poles of Jones' factors and Ekman's entire factor we have considerable evidence for invariance. But then what about the negative poles of Jones' curves?

Jones thinks that the high negative loadings at the blue end of the spectrum mean that the long wave (red) receptor is insensitive to blue-green hues, but that it is sensitive to brightness. Since brightness depends on wave amplitude and hue on wave length, this would mean that the long wave (red) receptor could respond to changes in wave length only in the red end of the spectrum, but that it could respond to changes in wave amplitude thoughout the spectrum. He does not spell out the meaning of his explanation in physiological terms.

Pickford (1951) has interpreted Jones' red factors as red-green bipolar factors. This would require that the bipolar receptor for red and green gives one type of response for red and another for green. This of course, is not compatible with

JONES' BLUE FACTORS

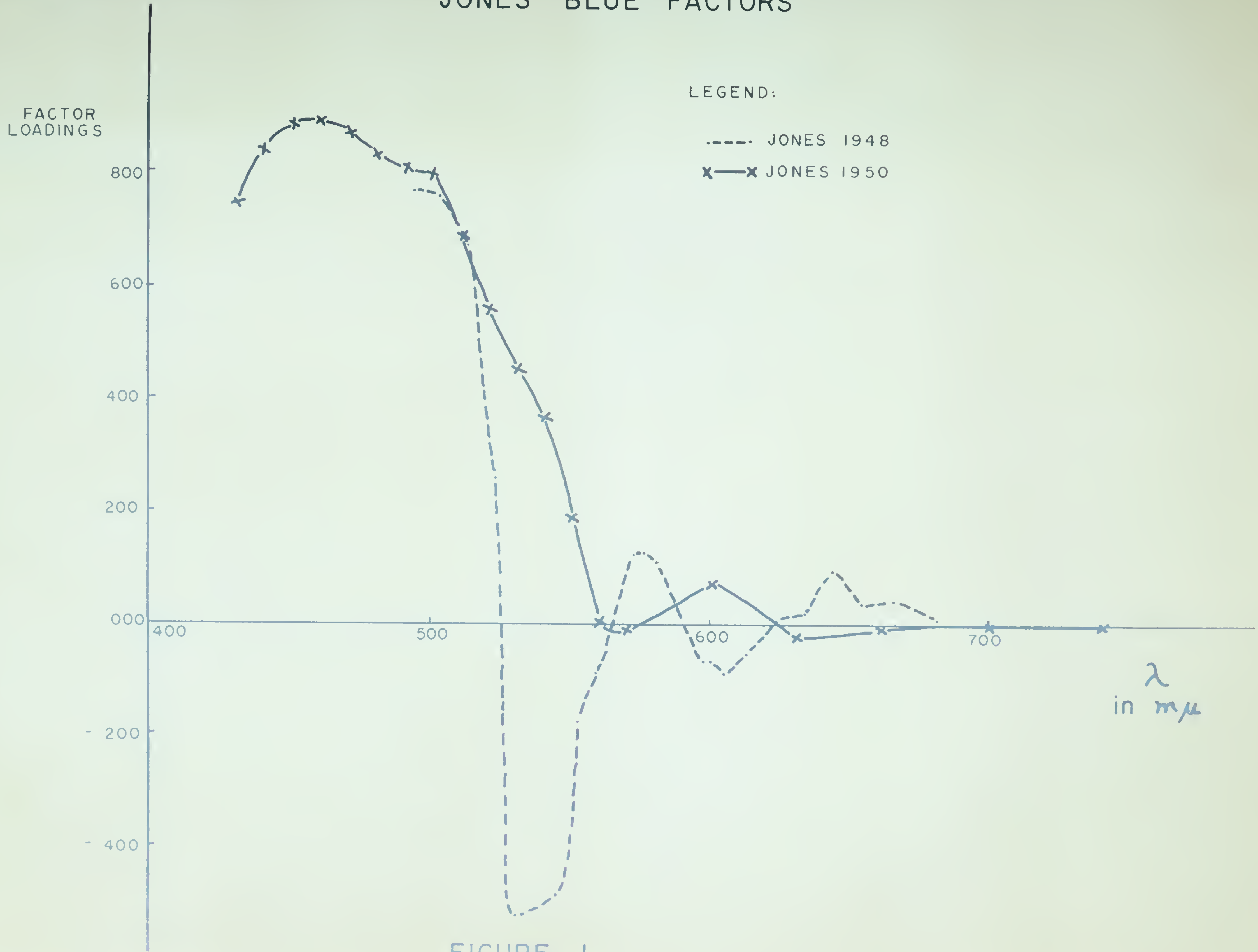


FIGURE I

JONES' YELLOW FACTORS

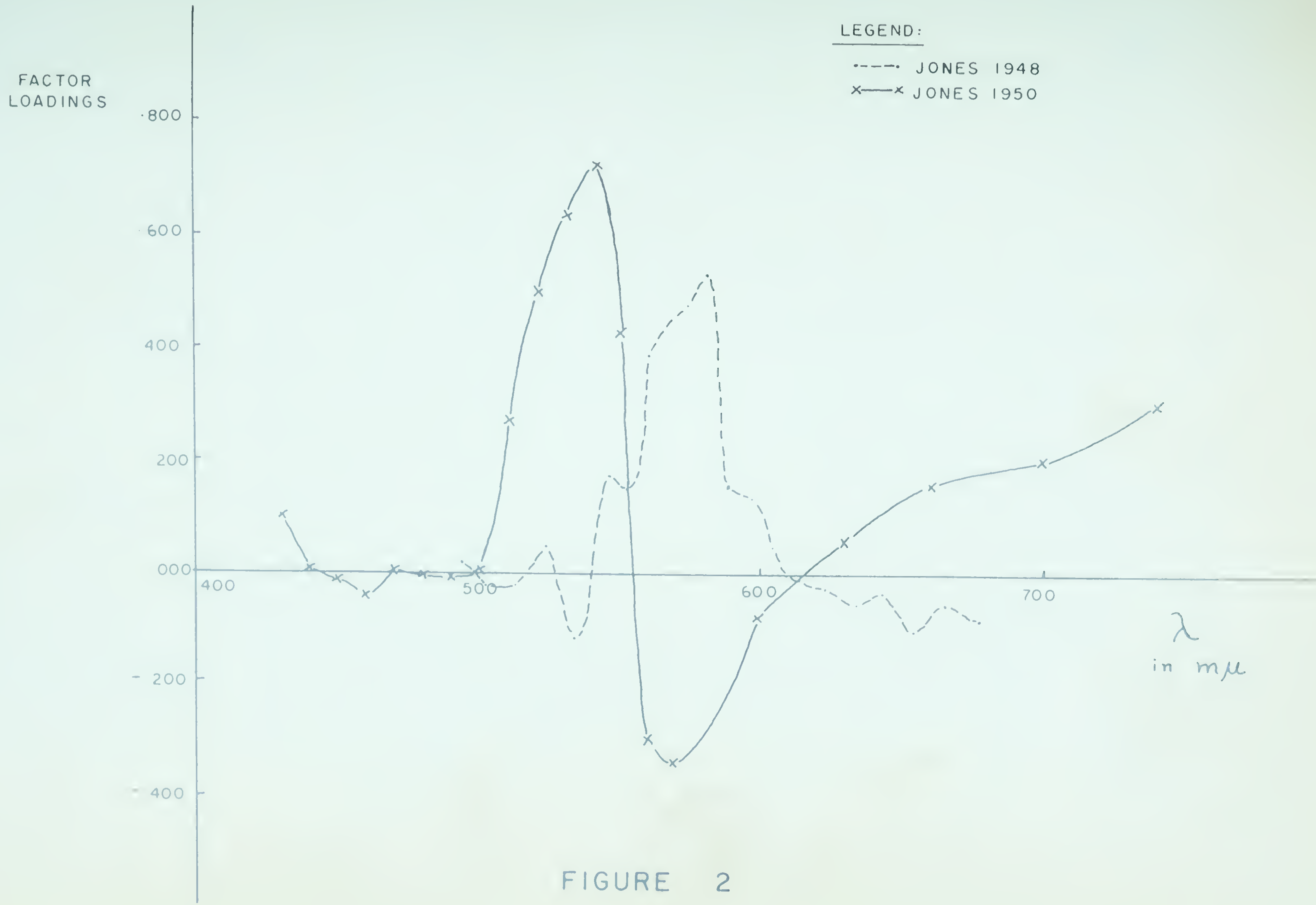
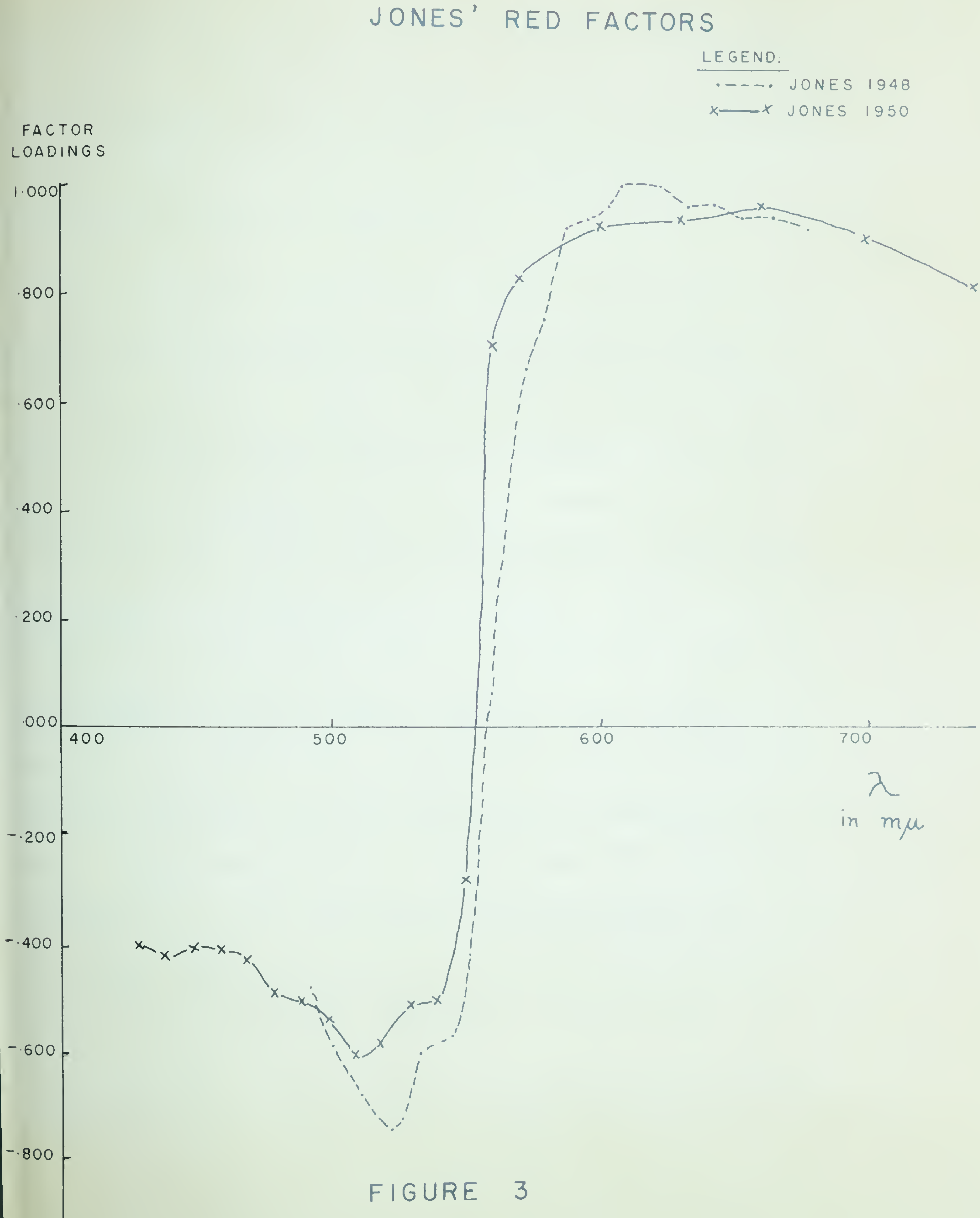


FIGURE 2



JONES' RED FACTORS
LEGEND:
JONES 1948
JONES 1950
FACTOR LOADINGS
1·000
·800
·600
·400
·200
·000
-·200
-·400
-·600
-·800
400
500
600
700
λ in mμ

FIGURE 3

the principle of "specific energies of nerves" and the "all or none law." However, if the "graded photopic response" which has been found by Svaetchin (1958) and others holds up under further research this could provide a possible physiological explanation for bipolar receptors.

The interpretation of the red factor is particularly important when we attempt to relate the factorial findings to the theories of color vision. If Pickford's interpretation is correct then the results would tend to support a four color theory such as Herring's, but if Jones' interpretation is correct then the results would tend to support a three color theory such as the Younge-Helmholtz theory. However, Ekman, who obtained violet, blue, green, yellow and red factors believes that his results support Granit's theory (1943). He shows that his factors when plotted against the spectrum are similar to Granit's electrophysiologically determined modulator curves. Since Granit believes that his modulator mechanisms are important in color perception this may indicate that there is some correspondence between the statistically determined factors and the underlying mechanisms of color vision.

Pickford (1951) reports several factor analyses of color

vision data. He used Burt's methods and did not rotate. In one study he had subjects rate the degree of binocular color combination when different colored lights were presented to different eyes. In another study he intercorrelated subjects' color weaknesses as measured by an anomaloscope. He also carried out analyses on red-green blind, red anomalous and green anomalous subjects. His usual result is one general factor and two bipolar factors. According to Pickford the general factor represents brightness discrimination. The two bipolar factors, a red-green and a yellow-blue, represent two bipolar processes.

In normal subjects the red-green factor contributes more to the total variance than does the yellow-blue factor. However, in all classes of red-green defectives the yellow-blue contributes more to the total variance. This, says Pickford, accords with the observation that in normals the red-green distinction is much sharper than the yellow-blue distinction.

For the red-green blind subjects, red and green discrimination is combined in the second bipolar factor in opposition to yellow and blue. This indicates to Pickford that the red-green process is not entirely lost in red-green blindness, but is modified in some way so that yellow continues to be

discriminated, but red-green discrimination is lost.

Pickford believes that his results best support a four color theory such as Houstoun's (1932) with one unipolar receptor for brightness and two bipolar receptors, one for red-green and one for yellow-blue. He also relates his work to that of Granit (1943). He says that the action of the scotopic dominator is not tested by his experiments, but that the activity of the photopic dominator might be represented by the general factor. The red-green and yellow-blue bipolar factors might represent two sets of modulators which Pickford believes are bipolar in nature.

Thus factor analysis has not yet contributed very much to our understanding of color vision. There is some evidence for the invariance of a blue, a yellow, and a red factor. This would seem to support a three color theory until we remember that by the laws of additive color mixture they should be blue, green and red. The only way to resolve these three factors with color vision theory is to say that there must also be a green factor which has not yet been shown to be invariant. This, of course, would give support to a four color theory.

Whether factor analysis can make a significant contribution

to our knowledge of color perception is still in doubt. If it is to make a significant contribution, a program of research which includes at least the following steps must be carried out. Color blind and color weak subjects must be excluded from the initial analysis since they may be representative of a population which does not include the same factors as normal subjects. Only after the factors of normal color vision had been established would it be pertinent to study color blind and color weak subjects. In comparing results from different subjects the sensitivity measurements must be made on the same retinal elements (i.e. rods or cones). This could be accomplished by controlling the brightness of the stimulus and the retinal angle of presentation. And lastly the factors must be shown to be invariant over several different studies.

Method

After considering several sets of data, two sets of correlations based on two independent determinations of the photopic curve were chosen. These data were selected because of their amenability to exploration by factor analysis, and because of the possibility that the resulting factors may be related to psychophysiological knowledge in the field of color vision. The two sets of data were collected by Coblentz and Emerson (1918) and Gibson and Tyndall (1923), and the correlations were computed by Jones (1948 and 1950).

Coblentz and Emerson (1918) used the method of flicker photometery to determine the sensitivities for ninety-two subjects. In this method a source constantly emitting the same white light repeatedly alternates with a monochromatic light of wave length λ and radiance L. The sensitivity at wave length λ is the reciprocal of the energy L required to make the flicker disappear.

Gibson and Tyndall (1923) used the step by step method, which is similar to the method of limits, to determine the sensitivities for thirty-eight subjects.

In both studies sensitivity measurements were made at twenty different points along the spectrum. The first set

of data covers wave lengths from 493 mµ to 678 mµ, and the second set covers wave lengths from 430 mµ to 740 mµ. Thus the first is deficient in measurements at both ends of the spectrum; this could cause some problems in defining the blue and red factors for this particular set of data.

These two methods of photometry did not give exactly the same results (Gibson and Tyndall 1923). The main differences are: (1) there is a difference in the wave length of maximum visibility, 557 mµ in the first study and 553 mµ in the second; (2) on the red side of the maximum the measurements for the first study are higher out to about 640 mµ, but beyond 650 mµ they do not differ greatly; (3) on the blue side of the maximum the curves are almost coincident out to 490 mµ, where they diverge. These small differences should not cause any major differences in the factorial results of the two analyses. If any differences do occur they should occur in the yellow-green part of the spectrum.

The intercorrelations were factored on the I.B.M. 1620 at the University of Alberta Computing Centre. An attempt was made to use image analysis (Guttman 1953) in order to get the best possible estimate of the communalities, but this procedure lead to imaginary factors so the data were re-

factored using 1's in the diagonals. Using 1's for communalities was justified by Wrigley's conclusions, reported in Harman (1960), that when the analysis includes twenty or more variables there is very little change in results when 1's are substituted for communalities. Kaiser (1960) also believes that using 1's in the diagonals is quite acceptable from a practical point of view. Kaiser's criteria of accepting all factors with associated latent root greater or equal to one was used to determine the number of factors to be rotated. The factors were first analytically rotated to orthogonal simple structure by the varimax method and then to oblique simple structure by graphic methods.

Unfortunately, the points on the spectrum where the measurements were made were not quite the same in the two studies. Therefore twelve variables that were close together in terms of millimicrons were chosen and considered as the common variables. The mean of differences was 3.2 mμ, approximately the same as the average j.n.d. for vision. The differences ranged from one mμ to six mμ and spanned the spectrum from 493 mμ to 678 mμ, i.e., from blue-green to orange-red.

The loadings on the "common variables' were adjusted

to the same units of measurement by the method recommended by Tucker (1951). Only two techniques, Kaiser (1960) and Wrigley-Neuhaus (1955), were considered applicable to the case of two oblique studies. The Wrigley-Neuhaus transformations were computed on the oblique reference factors and Kaiser's transformation on the varimax factors.

Results

The four oblique simple structure factors, the transformation matrix from the varimax solution, and the intercorrelations between the reference factors for study one are shown in table 1. The same information for the five factors of study two is shown in table 2. Table 3 gives the matrices of factor similarity for both the Kaiser and Wrigley methods. Table 4 gives the transformations that maximize the relationships between the two sets of factors.

Three of the factors from study one were judged significant and named red, green, and blue. They are plotted against the spectrum in figure four. The four factors judged significant in study two are plotted against the spectrum in figure five. They were named red, blue, green, and yellow. In order to facilitate comparison the red, green, and blue factors from the two studies are plotted against the spectrum in figures six, seven, and eight. And to aid interpretation, the squares of the loadings of the factors common to the two studies are plotted in figures nine and ten.

Table 1

Study 1. Oblique Simple Structure

λ	R	B	G	NS*
493	-.061	.954	-.079	-.023
502	-.004	.969	-.032	-.005
512	.059	.952	.076	.051
523	.132	.681	.423	.159
534	.091	.436	.697	.143
546	-.007	.159	.828	-.018
552	-.056	.001	.666	-.282
559	.002	.004	.004	.936
573	.712	-.061	.105	.179
580	.856	-.028	-.099	.081
587	.897	.058	.032	-.029
596	.918	.025	-.019	-.019
604	.929	.014	.049	-.034
613	.938	.016	.032	-.025
623	.955	.003	-.002	-.068
633	.952	.014	-.027	-.075
643	.944	-.061	.081	-.068
659	.921	.027	.012	-.070
665	.918	.010	-.006	-.011
678	.887	.030	.010	.025

Transformation matrix from varimax to oblique simple structure

	R	B	G	NS*
R	.900	.119	.207	-.050
B	-.099	.991	-.505	.000
G	-.402	.000	.968	.000
NS*	-.139	-.060	.137	.995

Intercorrelations between reference vectors

	R	B	G	NS*
R	1.001			
B	.017	.999		
G	-.217	-.033	.999	
NS*	-.183	-.066	.126	.992

* non significant factor

Table 2

Study 2. Oblique Simple Structure

λ	R	B	G	Y	NS*
430	.037	.543	.017	.011	.114
440	-.074	.546	.352	.318	-.040
450	.064	.737	-.064	.033	.091
460	.045	.754	-.033	.053	.026
470	.059	.721	.037	.026	-.057
480	.004	.700	.011	-.007	-.065
490	-.015	.671	.046	-.019	-.104
500	-.055	.617	.121	.003	-.140
510	-.084	.402	.331	.008	-.057
520	-.030	.188	.523	-.025	-.054
530	.002	.020	.636	.014	.072
540	.019	-.038	.500	-.053	.348
550	-.009	.005	.001	.006	.880
560	.043	.011	.060	.773	.336
570	.197	-.011	.159	.690	.002
600	.559	.106	.081	.438	.025
630	.672	.017	.040	.256	.032
660	.784	.020	.056	.134	-.071
700	.798	.035	.009	.080	-.022
740	.846	-.014	-.027	-.122	.002

Transformation matrix from varimax solution to oblique simple structure

	R	B	G	Y	NS*
R	.959	.167	.078	-.030	-.039
B	.142	.848	-.080	.118	-.076
G	.073	-.357	.935	.196	-.037
Y	-.221	.051	.279	.973	.302
NS*	.079	.349	-.189	-.002	.949

Intercorrelations between the reference vectors

	R	B	G	Y	NS*
R	1.000				
B	.271	.999			
G	.055	-.440	1.000		
Y	-.213	.074	.443	1.000	
NS*	-.041	.296	-.127	.278	.999

* non significant factor

Table 3

Factor Similarity Matrices

Kaiser

Study Two

		R	B	G	Y	NS*
Study One	R	.868	-.152	-.105	.512	-.071
	B	-.305	.939	.655	.307	-.372
	G	.059	.038	.627	-.344	.784
	NS*	-.179	-.027	-.093	.753	-.206

Wrigley - Neuhaus

Study Two

		R	B	G	Y	NS*
Study One	R	.934	-.054	-.151	.276	-.175
	B	-.039	.778	.114	-.126	-.252
	G	-.022	.008	.938	.039	.005
	NS*	.048	.045	.036	.463	.039

* non significant factor

Table 4

Transformations to maximize relationships

Kaiser's K

	R	B	G	Y	NS*
R	.8780	.1475	.3259	.3011	.0983
B	-.1866	.9082	.2877	.1484	-.1902
G	-.2592	-.1378	.6601	.1142	.6812
NS*	-.3544	.0075	.1422	.9160	-.1170

Wrigley - Neuhaus

T_1

	R	B	G	Y	NS*
R	-.3435	.6612	-.4814	.0508	.4588
B	-.6429	-.4615	-.1049	.6022	.0073
G	-.0160	.0190	.7108	.1132	.6939
Y	-.5152	-.3073	-.0093	-.7886	.1343
NS*	-.5406	.5051	.5020	-.0001	-.5384

T_2

	R	B	G	NS*
R	8664	4602	-.0836	.1738
B	.4114	- 8458	-.3415	.0266
G	.2080	-.2686	.9304	.1302
NS*	.1916	.0256	.1032	-.9759

* non significant factor

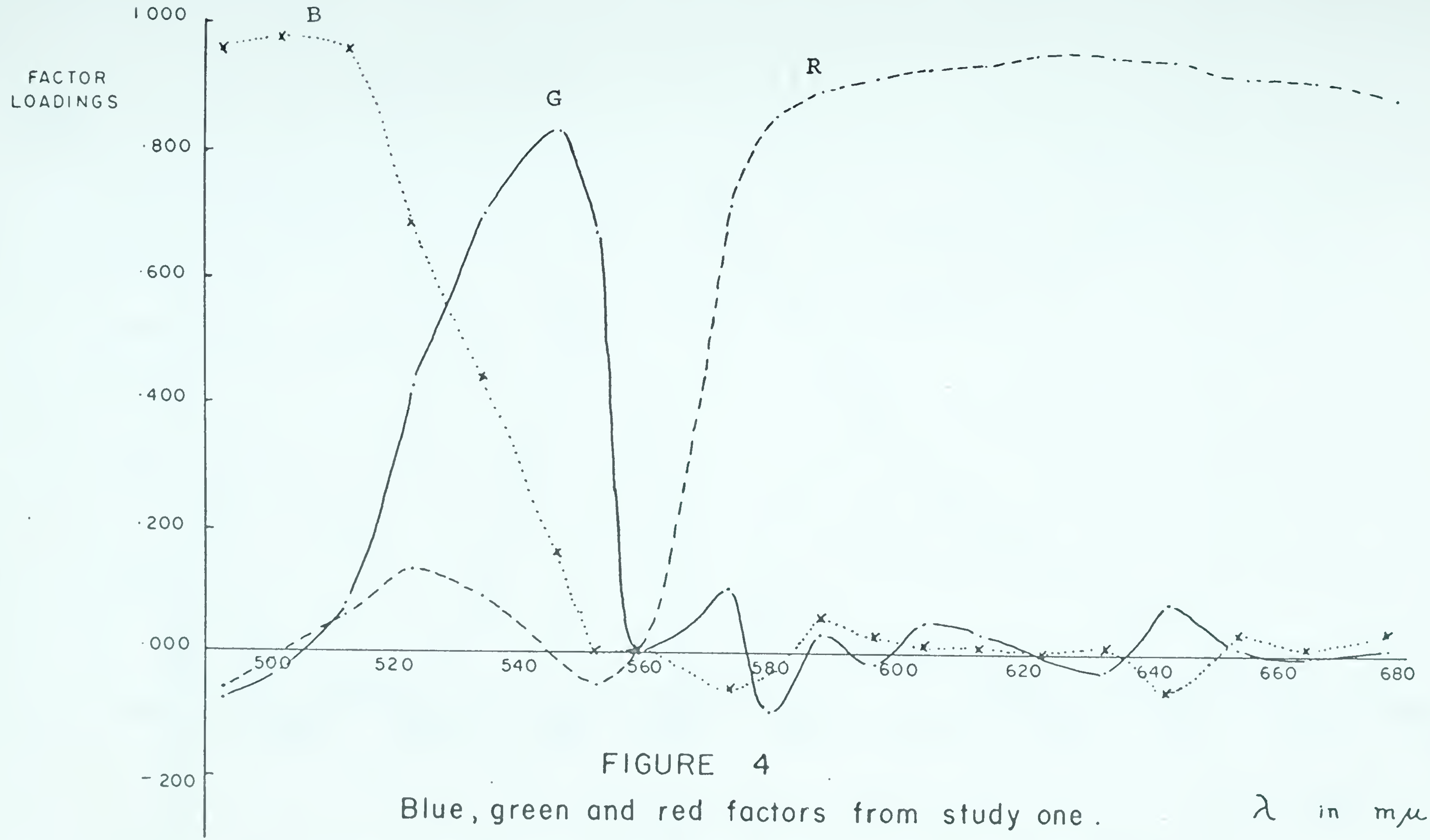


FIGURE 4

Blue, green and red factors from study one.

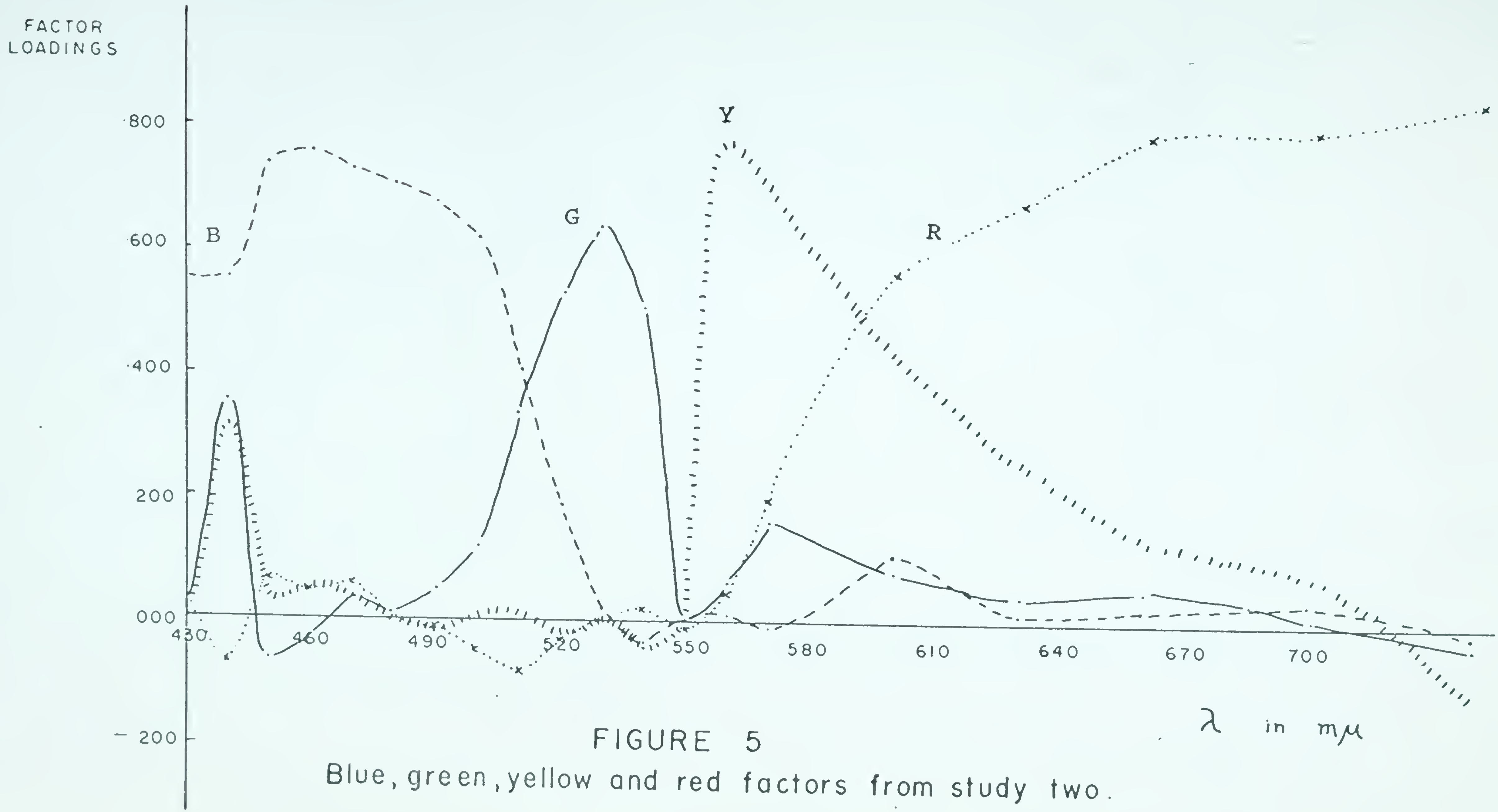


FIGURE 5

Blue, green, yellow and red factors from study two.

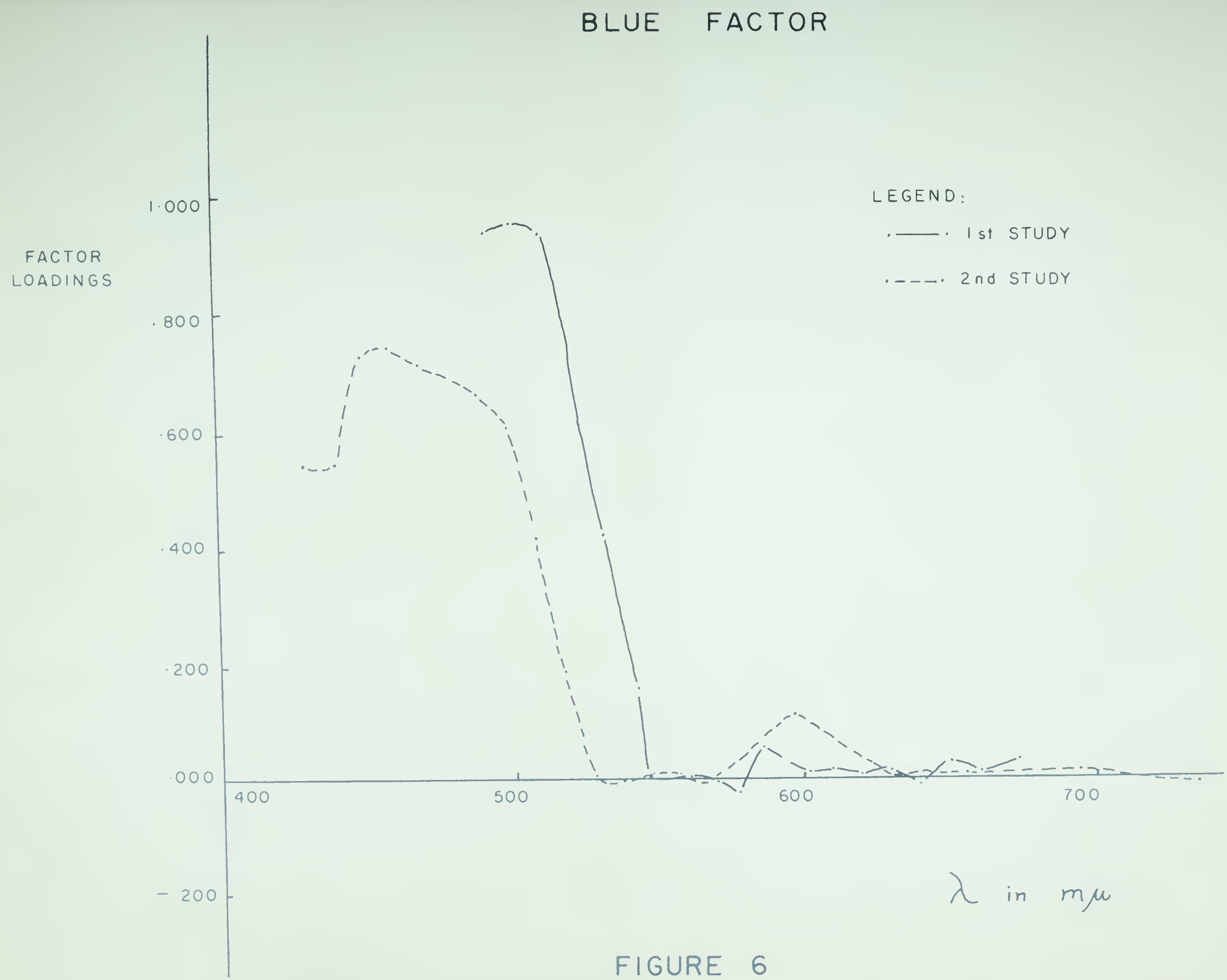


FIGURE 6

Blue factors from studies one and two.

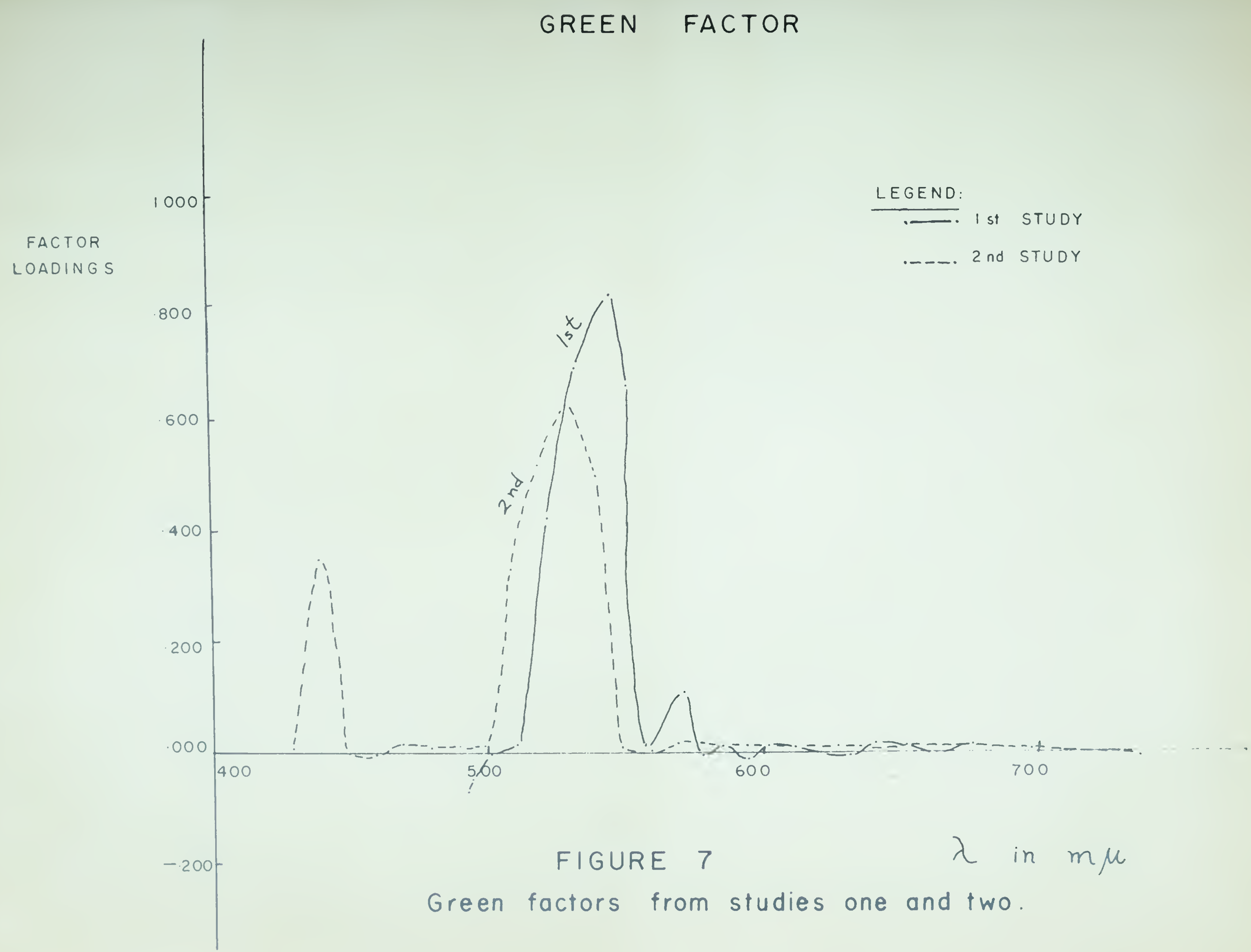


FIGURE 7

Green factors from studies one and two.

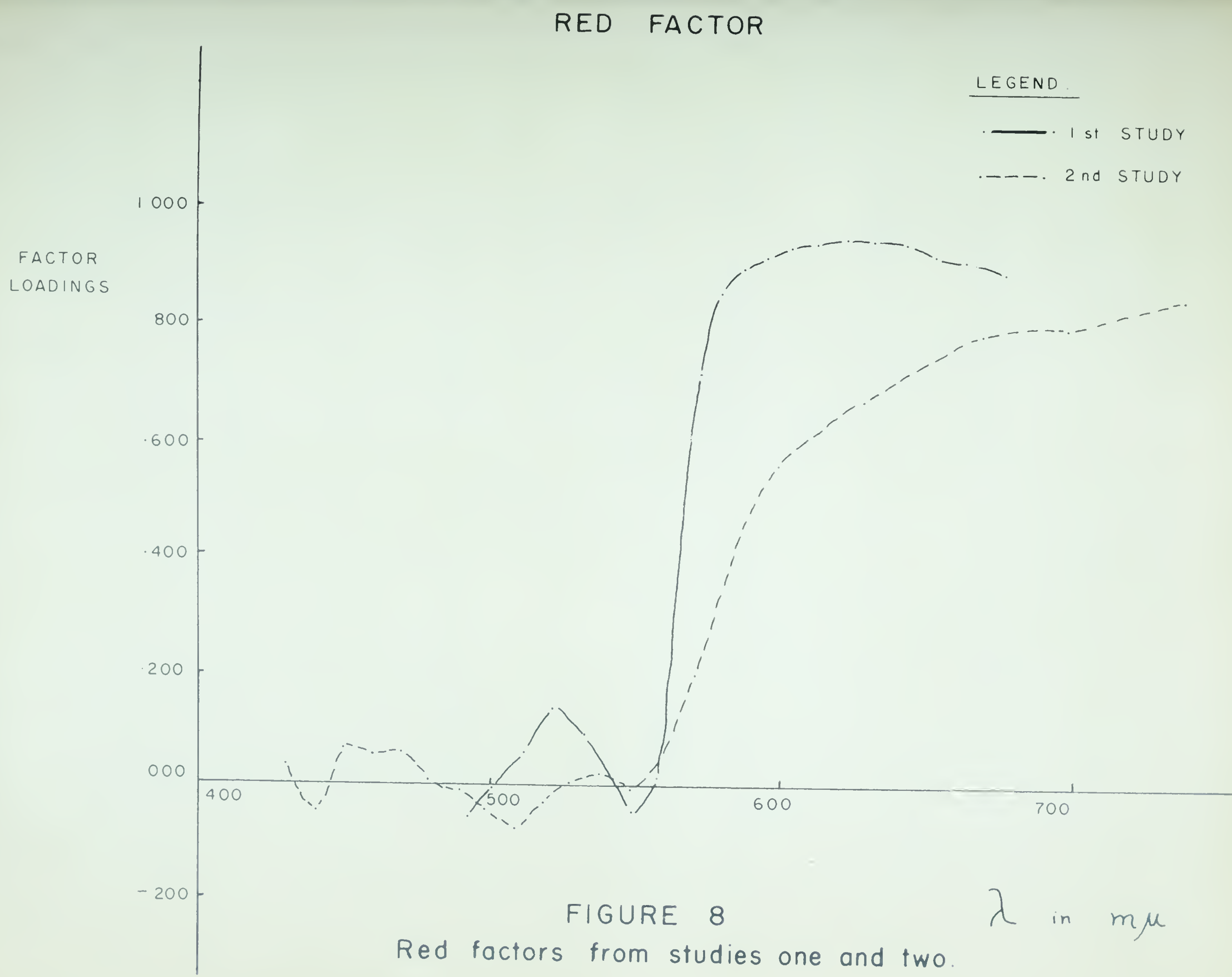


FIGURE 8
Red factors from studies one and two.

Discussion

Invariance of Factors

Both methods seem to indicate that there is a high degree of similarity between the red, green, and blue factors of the two studies. Both methods also indicate that there may be a relationship between the nonsignificant factor of study one and the yellow factor of study two. This means that the fourth factor of study one may actually be a yellow factor.

However, a careful study of table three shows that the relationship between the factors of the two studies is not simple. Kaiser's method indicates that there is a slightly higher relationship between the nonsignificant factor of study two and the green factor of study one than between the green factors of the two studies. His method also suggests that the green factor of study two is about equally related to both the blue and green factors of study one. Wrigley's method supports neither of these indications. The results of both methods taken together seem to suggest that there is a fairly high degree of similarity between the red, green, and blue factors of the two studies, and give some support to a common yellow factor.

Experience with these two methods has given some insight

into their properties. Wrigley's method is not completely objective. His transformations maximize the relationship between corresponding factors in the two studies. If a factor matrix had two similar factors, e.g. co-operative factors (Cattell 1962), it might be difficult to know which of them should be paired with a factor that is similar to both of them in the other study. This problem does not arise with Kaiser's method since he matches tests instead of factors. The difference in the results produced by these two methods can be seen by an examination of the two factor similarity matrices. Wrigley's method shows a high relationship between the paired factors (high diagonal entries) and a low relationship between the non-paired factors (low off diagonal entries). Kaiser's method, however, shows several fairly high relationships between non paired factors, e.g. between the green factor of study two, and both the blue and green factors of study one.

Neither Kaiser's nor Wrigley's method can be judged superior, since under certain circumstances one may be more appropriate while under other circumstances the other may be more appropriate. For example, if the relationship between two sets of factors that are already fairly well understood is to be maximized, then Wrigley's method would be most useful.

But if all the relationships between two sets of factors that are not yet fully understood are required then Kaiser's method is most appropriate.

Since Kaiser's transformation is calculated on the original, arbitrary orthogonal factor matrices it is also arbitrary to a certain extent. He calculates it from the relation

$$F_1' \ F_2 \ K = U \lambda^{1/2} U'$$

where the right hand side gives the inner product of the maximally similar arbitrary factor matrices. K is therefore a transformation that tends to eliminate the differences between the original factorings. Suppose that two studies were factored by the same method, e.g. the principal axis method, and K computed. Further suppose that one of the studies was refactored using a different method, e.g. the square root method, and K computed. The two K's would not be quite the same and neither would the two L_{12}'s ($L_{12} = T_1' \ K \ T_2$), the factor similarity matrices. But the two K's would be the same if orthogonal rotations were being compared and the transformation K was computed on these rotated orthogonal factor matrices.

Another problem with Kaiser's method is that it tends to put extra factors into F_2 when the number of factors in the two studies is not the same. In this case r_2 is less than r_1 and F_1 will be n by r_1, F_2 will be n by r_2 and K will be r_2 by r_1. Now if we compute Q, the matrix of transformed factors, we get

$$ {}_nF_{r_2}\;{}_{r_2}K_{r_1} = {}_nQ_{r_1} $$

and K puts $(r_1 - r_2)$ extra factors into F_2. These extra factors will usually be small, but it is possible to think of cases where they may be appreciable. For example, if the $(r_1 - r_2)$ extra factors in F_1 make a fairly large contribution to the total variance of F_1 then the transformation K would make the extra factors of F_2 contribute appreciably to the variance of F_2.

Kaiser (1960) draws a distinction between matching factors and relating factors. In factor matching the two sets are rotated as close together as possible and then an attempt is made to pick pairs of factors which are a good match. In relating factors the problem is to find the relationship between two sets of factors regardless of the strengths of the relationships. This distinction needs to be clarified and related

to the problem of factorial invariance.

Suppose there are two n by r factor matrices, A_1 and A_2. It is always possible to find a transformation T that will carry A_1 into A_2. Setting

$$A_1 T = A_2$$

$$A_1' A_1 T = A_1' A_2$$

$$\text{and} \quad T = (A_1' A_1)^{-1} A_1' A_2$$

provides an expression for this transformation. A different procedure would be to rotate both studies to a principal axis solution. In either case it is possible to get two sets of factors which are a very good match. If these two sets are then rotated to simple structure they will be very similar and one might be tempted to conclude that simple structure gives invariant results. But such a conclusion would not be warranted. With this type of procedure it is almost impossible not to get positive results.

If the transformation to maximize similarity were computed after independent rotations then it would still be possible to get two sets of simple structure factors that are very similar. But in this case the transformation matrix can be used to give some indication of whether the factors

are invariant. If the transformation resembles an identity matrix then this is strong evidence for invariance. Ahmavaara (1954) uses it as a measure of relationship between two sets of factors, but this transformation actually gives a measure of how much one factor matrix must be changed in order to make the factors look invariant. As such it is not a very meaningful measure of factor similarity.

In relating factors one wishes to compare factors from two studies as they stand. Sometimes it may be advisable to make slight transformations in one or both sets in order to maximize certain relationships. Since neither set of factors has been substantially changed a strong relationship between the two sets can be taken as evidence for invariance. Only when two independently rotated sets of factors appear similar is there evidence for invariance. But the similarity of two sets of factors is not sufficient to establish invariance. An attempt will now be made to establish a condition for the invariance of a set of factors.

Suppose that there is some unique simple structure factor loading matrix for a domain. Let it be defined as

$$\mathcal{U} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} V_i$$

where

V_i = an oblique simple structure factor loading matrix

N = the number of independently rotated simple structure matrices

Any particular V_i should be an approximation to $\mathcal{U}$. There should be a transformation E_i which will carry each V_i into $\mathcal{U}$. We have

$$\mathcal{U} = V_i E_i$$

where E_i represents all sources of variation that prevent any V_i from equaling $\mathcal{U}$. Note that E_i is not necessarily square and may add or subtract a factor from V_i. E_i is a function of $\mathcal{U}$ and can never be known since $\mathcal{U}$ is an unknown. It seems to follow that some kind of average, $\overline{U}$, of a number of V_i's should give a good approximation to and that $\overline{U}$ should approach $\mathcal{U}$ as the number of studies approaches infinity. More specifically:

$$U = \frac{1}{M} \sum_{i=1}^{M} V_i \ T_i$$

and

$$\lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} V_i \ T_i$$

where T_i is a transformation that carries each V_i, an in-

dependently rotated simple structure matrix, into $\overline{U}$, the best average of a number of V_i's, and M, is the number of studies averaged to get $\overline{U}$. This suggests that on the average the T_i's should be an identity matrix, i.e. on the average there is no transformation, and provide a condition for the existence of a unique $\mathcal{U}$. This condition may be formally stated as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} T_i \longrightarrow I$$

This may be used as a criterion for the existence of an invariant set of factors. If several sets of factors are transformed to maximum similarity then the average of these transformations should approximate an identity matrix. From table four it can be seen that the transformations to maximize similarity resemble an identity matrix. If a large number of studies had been performed and the resulting factor matrices transformed to maximize similarity among them, then the average of these transformations would probably have been an identity matrix.

Interpretation of Invariant Factors

During the past half century there has been considerable controversy over the nature of factors. Some writers believe that they are descriptive statistical concepts while other writers are willing to credit them with a considerable degree of reality. For a review of the relevant literature see Coan (1964). None of these writers have, however, focused on the nature of invariant factors. Starting with Hartley's (1954) distinction between descriptive factors and inferential factors an attempt will be made to determine the nature of invariant factors.

Descriptive factor analysis, according to Hartley (1954), aims at converting the information in a correlation matrix into a more usable form in much the same way as a standard deviation converts the information in a frequency distribution into a more comprehensible form. All the data that the factor analyst wishes to deal with are included in the correlation matrix, and he does not wish to make any generalization about the kind of data not included in the correlation matrix. In other words, descriptive factor analysis is a technique, analogous to the drawings of the draftsman, for parsimoniously representing a mass of data. And as such any particular solution cannot be judged as right or wrong. Different solutions are

merely different ways of representing the same set of data; one solution may be more useful in one situation while a different solution may be more useful in another situation.

But inferential factor analysis, continues Hartley, is that form of factor analysis in which the factor matrix has empirical meaning in addition to the kind of information expressed in the correlation matrix. A factor is a dimension along which individual differences lie and therefore must correspond to something real. We are thus justified, concludes Hartley, in calling one solution right, and another wrong. Thurstone's concept of simple structure could be considered as an attempt to provide a criterion for an inferential factor analysis.

The delineation of these two types of factors now makes possible a closer understanding of the problem of invariance. Since descriptive factors are defined only in terms of their antecedents, the observed correlations, invariance can be obtained by picking a standard set of factors and marker variables and then rotating all other studies in a domain in terms of the standard set. Young and Householder (1940) have described a procedure that could be used to accomplish this end. No such simple solution presents itself in the case of inferential factors since every study must be independently rotated to

what is regarded as the correct solution, i.e. simple structure.

Another, and perhaps more fruitful distinction could be made between factors as intervening variables and hypothetical constructs. MacCorquodale and Meehl (1948) have attempted to differentiate between these two type of constructs as follows:

> "Concepts of the first sort (intervening variables) seem to be identifiable by three characteristics. First, the statement of such concepts does not contain any words which are not reducible to the empirical laws. Second, the validity of the empirical laws is both necessary and sufficient for the "correctness" of the statement about the concept. Third, the quantitative expression of the concept can be obtained without mediate interference by suitable groupings of terms in the quantitative empirical laws."

In terms of factor analysis these three characteristics seem to mean respectively: (1) that factors have no meaning other than the kind that is expressed in the correlation matrix, (2) that the meaning of the correlation matrix is both necessary and sufficient to determine the meaning of the factors, and (3) that the factors are nothing more than the rearrangement of the data in the correlation matrix. Factors considered as intervening variables can have invariance in the same trivial sense

as Hartley's descriptive factors. That is, invariance can be achieved by defining a standard set of factors and marker variables and rotating all subsequent sets in terms of the standard. This fact should not be taken to mean that descriptive factors or factors as intervening variables have no use, for example they can function adequately as predictors in many types of multivariate prediction problems.

MacCorquodale and Meehl (1948) attribute the following three characteristics to hypothetical constructs:

> "Their formulation involves words not wholly reducible to words in the empirical laws; the validity of the empirical laws is not a sufficient condition for the truth of the concept, insomuch as it contains surplus meaning; and the quantitative form of the concept is not obtainable simply by grouping empirical terms and functions".

These three characteristics of hypothetical constructs can be respectively related to factors in the following way. Factors, as hypothetical constructs, have meaning other than the kind of meaning expressed by the correlation matrix. That is, if a factor possesses information other than the descriptive statistical information possessed by the correlation matrix then it meets the first criterion necessary for a hypothetical construct. Secondly, the validity of the correlation matrix is not sufficient to establish the validity of the factors.

If a factor is to have the status of a hypothetical construct then other, non-factorial, information must support the existence of the factor. And the third criterion might be interpreted as meaning that a rearrangement of the data in the correlation matrix, as in factor analysis without rotation, is not sufficient evidence for the existence of a factor if it is to be considered as a hypothetical construct.

In the introduction an invariant factor is defined as a factor that is independent of both the particular sample of individuals and tests used in the analysis and six sources of evidence for invariance are listed. Of these six the two most important are the stability of the loadings profile and the behavior of the factor as an independent psychological variable. If the loading profile is stable this means that some acceptable criterion for rotation must have been used. And if the factor functions as a stable independent variable then it must have greater meaning than the descriptive statistical information expressed in the correlation matrix. The author concludes that if a simple structure factor is invariant it has the status of a hypothetical construct.

Now that an argument which at least sounds plausible has been made for considering invariant factors hypothetical constructs, an attempt will be made to relate the obtained factors

to some non-factorial knowledge in the field of color vision. In figures nine and ten the squares of the loadings of the factors common to the two studies are plotted against the spectrum. These curves show the approximate amount of variance in absolute threshold measurements that is accounted for by each factor. As such it seems reasonable to suppose that they represent the action of receptor mechanisms.

Both these sets of curves resemble the fundamental sensation curves which have been proposed to account for color perception on the basis of three receptors. The fact that they are not exactly the same as any of the several sets that have been proposed should not trouble us too much because these curves, themselves, represent extensive transformations of the original data in order to account for certain facts of color vision. What is significant is that these two sets of curves resemble the fundamental sensation curves even though they were derived without any reference to color vision data. Additional evidence supporting the supposition that the factors represent the action of receptors can be obtained if the factors are compared with Granit's modulator curves which were determined by electrophysiological techniques. The highest loadings for the three factors common to both studies fall at the same points in the spectrum as Granit's

FIRST STUDY

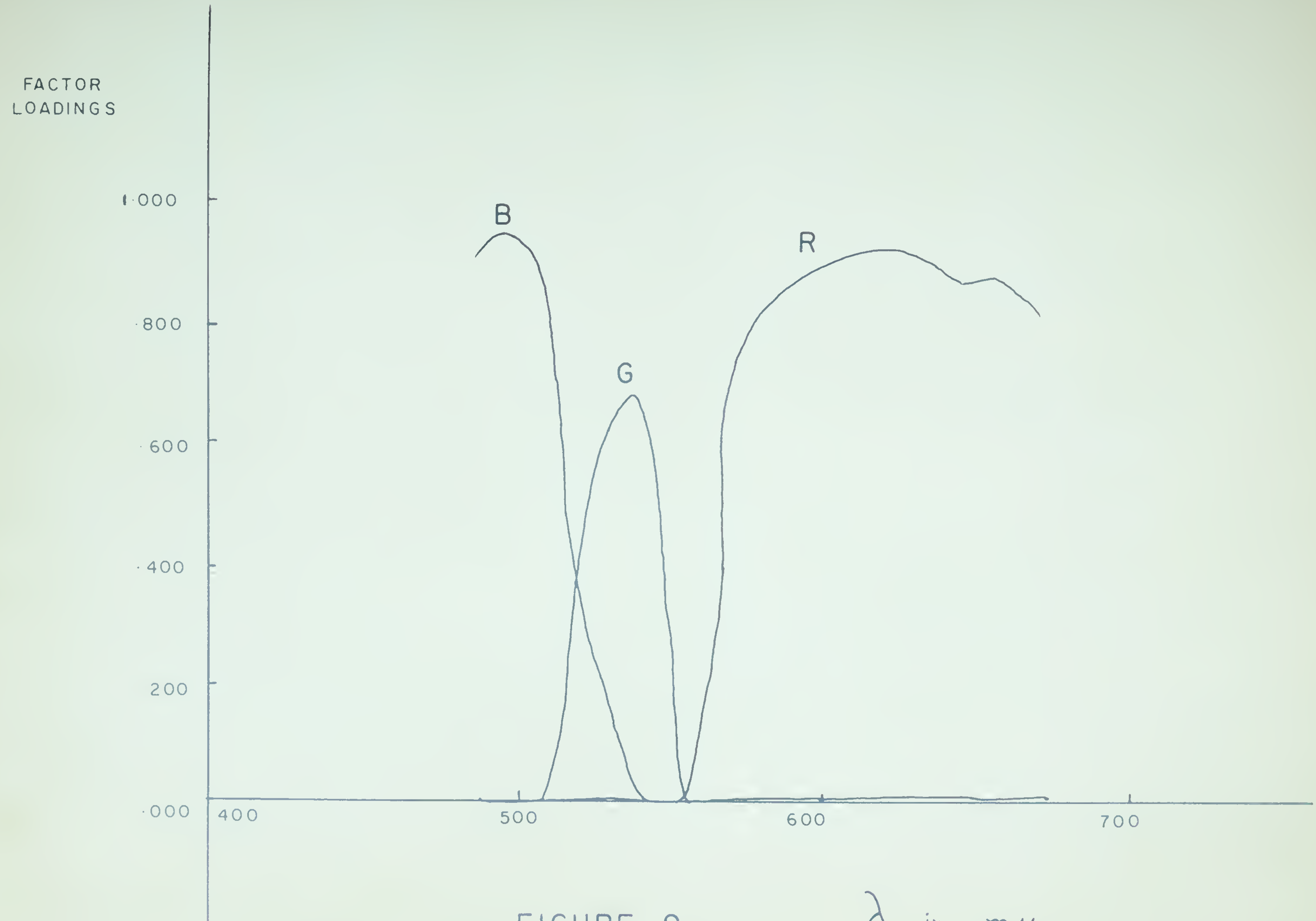


FIGURE 9

Squared factor loadings for blue, green and red factors of study one.

SECOND STUDY

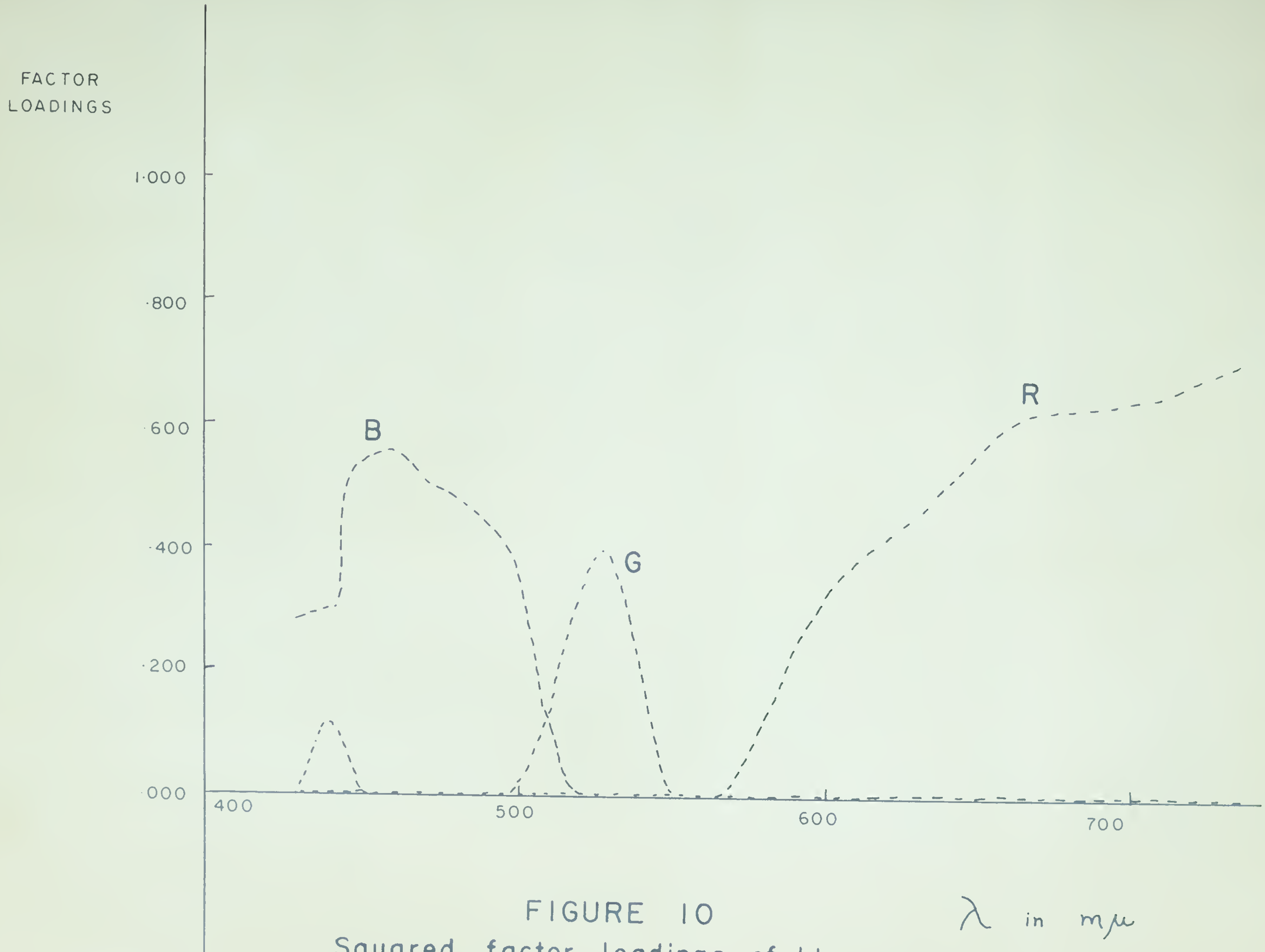


FIGURE 10

Squared factor loadings of blue, green and red factors of study two.

red, blue, and green modulator curves. This evidence seems to suggest that the three common factors represent the action of blue, green, and red receptors and supports the contention made earlier in the discussion that invariant factors may be considered as hypothetical constructs.

There are two other facts that must be briefly considered. The first is that there is some evidence for a yellow factor and the second is that the factors are correlated. Further studies would be necessary in order to determine if there is an invariant yellow factor. It is possible that the second study which included the significant yellow factor is more representative of the universe of psychological content because it includes a more comprehensive sample of variables and that further studies would verify the existence of a yellow factor. The correlation between the factors could be interpreted as meaning that the different receptors do not act independently of each other. This would fit in with recent evidence suggesting that the receptors do not act indepently of each other and that some kind of inhibitory mechanism may be involved in color perception (Milner 1963).

Conclusions

A comparison of Kaiser's (1960b) and Wrigley-Neuhaus' (1955) methods of relating factors shows that in general Kaiser's method is superior because it is more objective. If, however, the relationship between two fairly well understood sets of factors is to be maximized then Wrigley-Neuhaus' method may be more appropriate.

An invariant factor matrix is defined as

$$\mathcal{V} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} V_i$$

where the V_i refer to independently rotated factor matrices. From this it seems to follow that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} T_i \longrightarrow I$$

where T_i is a transformation that carries each V_i into the best average of them all must hold if $\mathcal{V}$ is to be unique.

The conclusion is drawn that an invariant factor has the status of a hypothetical construct. The fact that the three apparently invariant factors can be interpreted in terms of receptor mechanisms is used to support this conclusion.

The fact that red, green, and blue factors were common to the two studies lends support to a three receptor theory such as the Younge - Helmholtz theory. Electrophysiological techniques for investigating color vision are now advancing very rapidly (Milner 1963) and further factor analytic techniques would not now add greatly to our knowledge in this field.

Suggestions for Further Research

(1) Kaiser's method should be generalized so that the transformation K can be computed on the basis of two oblique factor matrices. If this were done then the case of same variables-different subjects would be essentially solved.

(2) The definition of an invariant factor matrix suggested in this paper should be investigated further. The author is now planning such an investigation.

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APPENDIX A

CORRELATION MATRICES

MATRIX I
Jones 1948

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1.																				
2.	.951																			
3.	.853	.911																		
4.	.531	.586	.716																	
5.	.292	.380	.528	.701																
6.	.073	.180	.257	.429	.651															
7.	.041	.023	.026	.208	.303	.405														
8.	.072	.067	.060	.041	-.043	-.196	-.241													
9.	-.076	-.184	-.290	-.432	-.505	-.358	-.240	.225												
10.	-.091	-.226	-.310	-.497	-.598	-.550	-.376	.192	.764											
11.	-.116	-.236	-.342	-.543	-.574	-.461	-.290	.092	.783	.896										
12.	-.107	-.212	-.339	-.522	-.589	-.511	-.349	.115	.763	.869	.949									
13.	-.065	-.199	-.307	-.490	-.572	-.466	-.312	.100	.785	.871	.933	.944								
14.	-.068	-.196	-.309	-.504	-.578	-.490	-.319	.128	.752	.844	.910	.932	.950							
15.	-.092	-.209	-.320	-.533	-.596	-.521	-.318	.099	.723	.840	.891	.919	.944	.957						
16.	-.096	-.202	-.301	-.522	-.585	-.510	-.368	.086	.691	.811	.864	.896	.926	.945	.968					
17.	-.157	-.264	-.377	-.565	-.613	-.519	-.320	.103	.689	.792	.861	.890	.923	.944	.963	.964				
18.	-.078	-.190	-.298	-.480	-.534	-.486	-.337	.089	.658	.757	.811	.844	.895	.929	.943	.957	.957			
19.	-.105	-.219	-.327	-.502	-.557	-.522	-.340	.156	.655	.764	.822	.846	.881	.924	.948	.951	.961	.955		
20.	-.077	-.175	-.264	-.475	-.495	-.505	-.343	.188	.629	.718	.794	.818	.847	.897	.925	.934	.939	.937	.961	

APPENDIX A

CORRELATION MATRICES

MATRIX II

Jones 1950

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1.																				
2.	.953																			
3.	.887	.969																		
4.	.809	.921	.973																	
5.	.762	.883	.944	.980																
6.	.753	.872	.924	.964	.981															
7.	.726	.838	.890	.938	.962	.984														
8.	.736	.843	.881	.926	.953	.973	.984													
9.	.719	.784	.808	.835	.870	.877	.889	.934												
10.	.639	.665	.680	.704	.754	.760	.771	.817	.941											
11.	.533	.532	.545	.564	.615	.620	.631	.672	.836	.951										
12.	.540	.504	.500	.481	.505	.515	.510	.537	.722	.850	.937									
13.	.370	.322	.318	.252	.194	.204	.177	.159	.302	.346	.451	.700								
14.	-.298	-.314	-.301	-.313	-.367	-.422	-.443	-.453	-.506	-.562	-.489	-.464	-.100							
15.	-.439	-.435	-.415	-.384	-.409	-.460	-.453	-.465	-.553	-.582	-.529	-.590	-.402	.875						
16.	-.388	-.382	-.332	-.317	-.331	-.381	-.381	-.420	-.498	-.504	-.477	-.500	-.298	.673	.848					
17.	-.406	-.421	-.397	-.413	-.403	-.469	-.492	-.530	-.571	-.532	-.488	-.480	-.256	.605	.729	.910				
18.	-.360	-.380	-.367	-.378	-.377	-.441	-.460	-.501	-.560	-.501	-.437	-.435	-.304	.512	.684	.867	.935			
19.	-.295	-.324	-.330	-.350	-.362	-.421	-.435	-.471	-.533	-.478	-.414	-.403	-.288	.430	.597	.785	.876	.966		
20.	-.310	-.328	-.331	-.352	-.356	-.408	-.415	-.437	-.461	-.425	-.339	-.287	-.135	.337	.454	.650	.754	.857	.918	

APPENDIX B

UNROTATED PRINCIPAL FACTORS

MATRIX I

	I	II	III	IV	V	VI	h^2
1.	.211340	-.917247	.002999	.184367	.179047	.063869	.956144
2.	.332881	-.896723	-.021832	.167928	.097453	.021058	.9535389
3.	.449083	-.851648	-.026628	.084222	-.022600	-.031694	.936297
4.	.639002	-.542371	.067763	-.190843	-.124396	-.025101	.759607
5.	.694335	-.292362	.263039	-.363786	-.285874	-.097294	.860312
6.	.594491	-.028481	.510990	-.383439	-.117848	-.284542	.857219
7.	.386467	.119412	.634632	-.155751	-.433362	.463200	.992989
8.	-.142921	-.200689	-.648818	-.659471	.096990	.236303	.981815
9.	-.775303	-.091815	.006949	-.231929	.364369	-.248850	.858055
10.	-.882753	-.110816	-.080935	-.041503	.211590	-.194172	.882279
11.	-.923537	-.083955	.110151	-.051356	.163911	-.173281	.931633
12.	-.939729	-.108577	.061681	-.024168	.089658	-.134627	.925431
13.	-.948863	-.150765	.128925	-.047626	.061303	-.097599	.955244
14.	-.958949	-.154409	.109594	-.044542	-.010908	-.001438	.957541
15.	-.967197	-.132955	.111973	.003599	-.055436	.052769	.971555
16.	-.957695	-.139957	.097320	.025609	-.140678	.045496	.968755
17.	-.964544	-.068250	.104178	-.001571	-.122332	.110489	.973031
18.	-.929615	-.152475	.123647	.001089	-.201679	.115784	.956802
19.	-.939027	-.125873	.066391	-.034161	-.190500	.167466	.967526
20.	-.908996	-.169054	.053552	-.062346	-.235648	.195341	.955296

EIGEN VALUES

	I	II	III	IV	V	VI
	12.062313	2.9979233	1,2660616	.91294496	.73700249	.62096057

APPENDIX B

UNROTATED PRINCIPAL FACTORS

MATRIX II

	I	II	III	IV	V	VI	h^2
1.	-.791832	-.320421	-.007852	.178199	.319358	.336473	.976687
2.	-.847699	-.383793	.096449	.129626	.266137	.163525	.989565
3.	-.859298	-.426101	.118656	.116153	.185345	.002491	.981884
4.	-.866856	-.428277	.169291	.055510	.062129	-.093607	.979223
5.	-.878454	-.412334	.153895	-.043758	-.027403	-.114853	.981241
6.	-.901756	-.342380	.171977	-.060952	-.026610	-.148870	.986549
7.	-.898238	-.315594	.178801	-.094784	-.079058	-.153513	.977201
8.	-.917848	-.279448	.167980	-.102412	-.125954	-.072540	.980368
9.	-.938641	-.142740	-.069333	-.071508	-.213818	.024493	.957660
10.	-.882530	-.057026	-.291662	-.136327	-.296611	.109465	.985723
11.	-.782880	.005192	-.487517	-.067457	-.345951	.120270	.989298
12.	-.728789	.086427	-.645849	.095903	-.137748	.042128	.985574
13.	-.393443	.120727	-.620928	.550525	.248756	-.272151	.993947
14.	.595389	-.380849	.131958	.622724	-.203722	.075543	.951941
15.	.701691	-.472560	.191485	.290343	-.354369	.092075	.970703
16.	.672573	-.646621	-.026576	.070550	-.201223	-.113065	.929431
17.	.726211	-.597109	-.170049	-.025451	-.039104	-.063243	.919014
18.	.699880	-.636108	-.223678	-.168777	.034946	.022801	.974724
19.	.658431	-.627405	-.260960	-.240595	.152869	.074393	.982057
20.	.593926	-.525290	-.392314	-.260428	.218263	-.048385	.900383

EIGEN VALUES

12.131497	3.3429634	1.6516835	1.0626112	.85013530.35517267	

APPENDIX C

VARIMAX FACTORS

MATRIX I

T RANSFORMATION MATRIX

.91933	-.24870	.30326	.03201
.25807	.95589	-.01316	.13965
-.27066	-.00396	.75408	.59840
.12241	-.15623	-.58243	.78828

CONVERGENT DENORMALIZED MATRIX

	A	B	C	D
1.	.02066	.95816	-.02895	-.02580
2.	.10108	.96610	-.00151	-.00474
3.	.21059	.93882	.07826	.05409
4.	.40578	.64782	.36317	.16517
5.	.44715	.39635	.62464	.14796
6.	.35394	.11719	.78931	-.01857
7.	.19527	-.03984	.68491	-.28604
8.	.08829	.05069	-.14587	.94070
9.	.76672	-.14126	-.09358	.21631
10.	.82331	-.12041	-.30310	.12489
11.	.90680	-.15701	-.16599	.01586
12.	.91159	-.13345	-.22296	.02739
13.	.95195	-.09879	-.16081	.01182
14.	.95655	-.09741	-.18019	.02179
15.	.95335	-.11244	-.20922	-.02030
16.	.93976	-.10000	-.23012	-.02821
17.	.93273	-.17447	-.21213	-.02068
18.	.92730	-.08478	-.18730	-.02379
19.	.91791	-.11828	-.21315	.03484
20.	.90142	-.07399	-.19674	.06981

APPENDIX C

VARIMAX FACTORS

MATRIX II

TRANSFORMATION MATRIX

.74400	-.44026	.17360	.29557	.35176	-.10679
.58448	.74345	-.05851	-.30979	-.03768	-.06960
-.28747	.39341	.56979	.22483	.62175	-.02816
-.00503	.28178	-.59162	.72056	.10998	.19814
-.01941	-.13482	-.36509	-.47780	.60643	.50210
.14765	.03327	.39808	.13644	-.32926	.83162

CONVERGENT DENORMALIZED MATRIX

1.	.73156	-.12566	.21126	-.11462	.16886	.58256
2.	.85407	-.20366	-.01582	.13290	.42474	.14258
3.	.92629	-.11603	.19203	-.09982	.08808	.23623
4.	.95923	-.13399	.12175	-.08793	.10893	.08196
5.	.95502	-.12828	.05050	-.13142	.18139	.00023
6.	.94162	-.19158	.05200	-.17332	.17119	-.03395
7.	.92480	-.21001	.01175	-.17899	.20072	-.07328
8.	.90225	-.24813	-.03042	-.17046	.27165	-.03111
9.	.75372	-.28934	.6397	-.19483	.51350	.1097
10.	.58348	-.23662	.8353	-.25034	.72054	.02152
11.	.41446	-.18837	.19990	-.20952	.83551	-00993
12.	.29762	-.17788	.48895	-.24709	.74809	.07478
13.	.09144	-.13122	.95402	-.05259	.22800	.05875
14.	.19441	.28790	.06313	.87999	-.22590	.04312
15.	-.20982	.45226	-.25282	.78794	-.17040	-.09126
16.	-.11695	.74405	-.12617	.52469	-.16738	-.20715
17.	-.23174	.83454	-.06827	.35311	-.16653	-.10847
18.	-.21676	.92055	-.12744	.21844	-.12621	-.02109
19.	-.20691	.91674	-.05930	.16611	-.14575	.09146
20.	-.23757	.91077	.03457	-.04846	-.10444	.00193

APPENDIX D

STANDARD DEVIATIONS AND ADJUSTMENTS
TO COMMON UNIT OF MEASUREMENT

	σ_1	σ_2	d_1	d_2
1.	55.191	41.221	1.145	.855
2.	60.490	53.960	1.057	.943
3.	53.540	67.402	.885	1.115
4.	40.534	39.357	1.015	.985
5.	31.407	51.284	.760	1.240
6.	23.865	31.983	.855	1.145
7.	32.819	14.957	1.374	.626
8.	56.300	21.693	1.444	.556
9.	33.270	47.687	.822	1.178
10.	60.556	94.657	.780	1.220
11.	45.519	69.915	.789	1.211
12.	9.691	19.235	.670	1.330

APPENDIX E

VARIMAX FACTORS FOR COMMON VARIABLES
ADJUSTED FOR COMMON UNIT OF MEASUREMENT

A	B	C	D	A	B	C	D	E
.024	1.097	-.033	-.030	.791	-.179	.010	-.153	.172
.107	1.021	-.001	-.005	.771	-.212	-.025	-.145	.233
.186	.831	.069	.048	.841	-.322	.071	-.217	.573
.412	.658	.368	.167	.574	-.233	-082	-.246	.710
.340	.301	.475	.112	.513	-.233	.248	-.260	1.035
.303	.100	.674	-.015	.341	-.204	.560	-.283	.856
.268	-.055	.941	-.393	.057	-.082	.597	-.032	.143
.127	.074	-.211	1.359	-.108	.160	.035	.489	-.126
.630	-.116	-.077	.177	-.247	.532	-.298	.928	-.200
.711	-.104	-.174	-021	-.143	.908	-.154	.641	-.204
.742	-.079	-.181	-.022	-.281	1.011	-.082	.427	-.201
.615	-.079	-.143	.023	-.289	1.224	-.169	.290	-.168

APPENDIX F

OBLIQUE FACTORS FOR COMMON VARIABLES ADJUSTED FOR COMMON UNIT OF MEASUREMENT

A	B	C	D		A	B	C	D	E
-.070	1.092	-.090	-.026		.574	-.013	-.089	-.016	.039
-.004	1.024	-.034	-.005		.582	-.052	-.132	.003	.114
-.050	.842	.067	.045		.448	-.094	-.063	.009	.369
.134	.691	.429	.159		.185	-.029	-.053	-.025	.515
.069	.331	.530	.109		.025	-.002	.089	.017	.789
-.006	.136	.708	-.015		-.043	.022	.298	-.061	.573
-.077	.001	.915	-.387		.003	-.006	.551	.004	.001
.003	.006	.006	1.352		.006	.024	.187	.430	-.033
.585	-.050	.086	.147		-.013	.232	.002	.813	.187
.716	.020	-.015	-.015		.129	.682	.031	.534	.099
.751	-.011	-.021	-.059		.021	.814	-039	.310	.048
.615	.007	-.004	-.007		.027	1.043	-.094	.356	.074